

**CRACKING OF A GRADED HALF PLANE  
DUE TO SLIDING CONTACT**

**Fazil Erdogan  
and  
Serkan Dag**

**Lehigh University  
Bethlehem, PA 18015**

**March 2001**

**AIR FORCE OFFICE OF SCIENTIFIC RESEARCH  
GRANT F49620-98-1-0028**

**DISTRIBUTION STATEMENT A  
Approved for Public Release  
Distribution Unlimited**

**20010419 102**

**DTIC CONVERSATION RECORD FOR DISTRIBUTION STATEMENT REQUEST**

DTIC Personnel Making Call <i>JACK RIKE</i>	Date <i>17 April 02</i>	Time <i>1425</i>
Authorizing Official <i>F. Erdogan</i>	Phone	
Agency <i>LEHIGH Univ. MECH. Engineering Mechanic</i>		
Title		
Internet Document URL (if applicable)		
<b>Distribution Statement</b> (Please check one box)		
<input checked="" type="checkbox"/> DISTRIBUTION STATEMENT A: Approved for public release. Distribution is unlimited.		
<input type="checkbox"/> DISTRIBUTION STATEMENT B: Distribution authorized to U.S. Government Agencies only.		
<input type="checkbox"/> DISTRIBUTION STATEMENT C: Distribution authorized to U.S. Government Agencies and their contractors.		
<input type="checkbox"/> DISTRIBUTION STATEMENT D: Distribution authorized to U.S. Department of Defense (DoD) and U.S DoD contractors only.		
<input type="checkbox"/> DISTRIBUTION STATEMENT E: Distribution authorized to U.S. Department of Defense (DoD) components only.		
<input type="checkbox"/> DISTRIBUTION STATEMENT F: Further dissemination only as directed by the controlling DoD office indicated below or by higher authority.		
<input type="checkbox"/> DISTRIBUTION STATEMENT X: Distribution authorized to U.S. Government agencies and private individuals or enterprises eligible to obtain export-controlled technical data in accordance with DoD Directive 5230.25, Withholding of Unclassified Technical Data from Public Disclosure, 6 Nov 84.		
Reason for the above identified distribution statement (in accordance with DoD Directive 5230.24)		
_____ _____ _____ _____		
Controlling Office	Date of Distribution Statement Determination	
	AQ Number (For DTIC-OCA Use Only)	

**CRACKING OF A GRADED HALF PLANE  
DUE TO SLIDING CONTACT**

Serkan Dag and Fazil Erdogan

*ME-MECH. Department, Lehigh University, Bethlehem, PA 18015*

**Abstract**

In this report the initiation and subcritical growth of surface cracks in graded materials due to sliding contact are considered. After a brief introduction the general coupled crack/contact problem for a semi-infinite graded medium subjected to a sliding rigid stamp of arbitrary profile is formulated. Solving the problem in the absence of any cracks, the complete stress state on the surface of the medium is evaluated and critical stress that would cause surface crack initiation is identified. The coupled problem is then solved, stress intensity factors are calculated and some results are presented.

**1. Introduction**

Graded materials, also known as *functionally graded materials* (FGMs) are multiphase composites with continuously varying volume fractions and, as a result, thermomechanical properties. Used as coatings and interfacial zones they reduce the residual and thermal stresses resulting from the material property mismatch, increase the bonding strength, improve surface properties and provide protection against severe thermal and chemical environments. Many of the present and potential applications of FGMs involve contact problems. These are mostly load transfer problems in deformable solids, generally in the presence of friction as in, for example, bearings, gears, cams, machine tools and abradable seals in gas turbines. In such applications the concept of

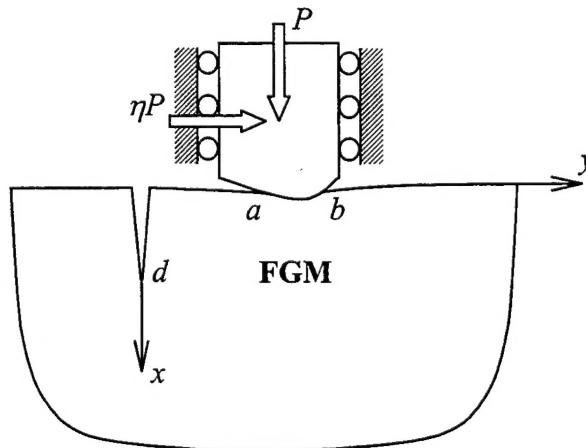
material property grading appears to be ideally suited to improve the surface properties and wear-resistance of the components that are in contact.

From the standpoint of failure mechanics an important aspect of contact problems is the surface cracking which is caused by friction forces and which invariably leads to fretting fatigue. In most applications material property grading near the surface is used as a substitute for homogeneous ceramic coatings. In both cases that is, in both homogeneous and graded coatings the surface of the medium consists of 100% ceramic which is generally a brittle solid. Hence, the "maximum tensile stress" criterion may be used for crack initiation on the surface. Once the crack is initiated, its subcritical growth under repeated loading by a sliding stamp is controlled by stress intensity factors at the crack tip. The main objective of this study is, therefore, the evaluation of peak tensile stresses on the surface for the purpose of studying crack initiation and the stress intensity factors for modeling the subcritical crack growth. Specifically, the objective is the examination of the influence of friction coefficient and material nonhomogeneity parameters on the peak surface stresses and stress intensity factors. The problem is considered under the assumptions of plane strain, Coulomb friction and linear nonhomogeneous elasticity.

Studies in contact mechanics in elastic solids were originated by Hertz [1]. The technical literature on the subject is very extensive. A thorough description of the underlying solid mechanics problems in homogeneous materials may be found, for example, in Johnson [2]. Some sample solutions for frictionless contact problems in a semi-infinite graded medium are given in [3]-[5]. Details of the analysis of homogeneous substrates with FGM coatings having positive or negative curvatures and extensive results regarding the stress distribution under plane strain conditions and sliding contact are discussed in Guler [6].

## 2. Formulation

The coupled crack/contact problem for a nonhomogeneous half-plane considered in this study is described in Figure 1. The half plane is in sliding contact with a rigid stamp of arbitrary profile. The normal and tangential forces transferred by the contact are  $P$  and  $\eta P$  respectively where  $\eta$  is the coefficient of friction, and contact area extends from  $y = a$  to  $y = b$ . The half-plane contains a surface crack of length  $d$  which is perpendicular to the surface. In this report, we will formulate the problem and reduce it to a system of singular integral equations. Solving the integral equations numerically we will examine the effects of material nonhomogeneity and friction on the stress intensity factors and contact stresses. Largely, for mathematical expediency it will be assumed that the elastic parameters of the medium may be approximated by



**Figure 1:** The general description of the crack/contact problem in a graded medium.

$$\mu(x) = \mu_0 \exp(\gamma x), \quad \kappa = \text{constant}, \quad (1a,b)$$

where  $\mu$  is the shear modulus,  $\gamma$  is the nonhomogeneity parameter,  $\kappa = 3 - 4\nu$  for plane strain and  $\kappa = (3 - \nu)/(1 + \nu)$  for generalized plane stress,  $\nu$  being the Poisson's ratio.

By using the Hooke's law

$$\sigma_{xx}(x, y) = \frac{\mu(x)}{\kappa - 1} \left\{ (\kappa + 1) \frac{\partial u}{\partial x} + (3 - \kappa) \frac{\partial v}{\partial y} \right\}, \quad (2a)$$

$$\sigma_{yy}(x, y) = \frac{\mu(x)}{\kappa - 1} \left\{ (\kappa + 1) \frac{\partial v}{\partial y} + (3 - \kappa) \frac{\partial u}{\partial x} \right\}, \quad (2b)$$

$$\sigma_{xy}(x, y) = \mu(x) \left\{ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right\}. \quad (2c)$$

The equilibrium conditions  $\sigma_{ij,j} = 0$  can be expressed as,

$$(\kappa + 1) \frac{\partial^2 u}{\partial x^2} + (\kappa - 1) \frac{\partial^2 u}{\partial y^2} + 2 \frac{\partial v}{\partial x \partial y} + \gamma(\kappa + 1) \frac{\partial u}{\partial x} + \gamma(3 - \kappa) \frac{\partial v}{\partial y} = 0, \quad (3a)$$

$$(\kappa + 1) \frac{\partial^2 v}{\partial y^2} + (\kappa - 1) \frac{\partial^2 v}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + \gamma(\kappa - 1) \frac{\partial v}{\partial x} + \gamma(\kappa - 1) \frac{\partial u}{\partial y} = 0. \quad (3b)$$

In previous studies (e.g. Delale and Erdogan [7]) it was shown that the stress intensity factors in graded materials are not significantly influenced by the variation in  $\nu$ . Thus, in this study too, the Poisson's ratio will be assumed to be constant. Following boundary conditions must be satisfied in the solution of the problem

$$\sigma_{xx}(0, y) = 0, \quad \sigma_{xy}(0, y) = 0, \quad -\infty < y < a, \quad b < y < \infty, \quad (4a,b)$$

$$\sigma_{xy}(0, y) = \eta \sigma_{xy}(0, y), \quad \frac{4\mu_0}{\kappa + 1} \frac{\partial}{\partial y} u(0, y) = f(y), \quad a < y < b, \quad (5a,b)$$

$$\sigma_{yy}(x, 0) = 0, \quad \sigma_{xy}(x, 0) = 0, \quad 0 < x < d, \quad (6a,b)$$

$$\int_a^b \sigma_{xx}(0, y) dy = -P, \quad (7)$$

$$\epsilon_{yy}(x, \pm\infty) = \epsilon_0, \quad (8)$$

where the known function  $f(y)$  defines the stamp profile. Note that, in addition to  $f(y)$  the external loads are described by the resultant force  $P$ , the remote strain  $\epsilon_0$  and the crack surface tractions given by (6a,b). We also observe that the unknown functions of the problem may be identified as follows,

$$\frac{2\mu_0}{\kappa+1} \frac{\partial}{\partial x} (v(x, 0^+) - v(x, 0^-)) = f_1(x), \quad 0 < x < d, \quad (9a)$$

$$\frac{2\mu_0}{\kappa+1} \frac{\partial}{\partial x} (u(x, 0^+) - u(x, 0^-)) = f_2(x), \quad 0 < x < d, \quad (9b)$$

$$\sigma_{xx}(0, y) = f_3(y), \quad a < y < b. \quad (9c)$$

In the following sections, we will derive the expressions for the stresses and displacements in the terms of the unknown functions  $f_j$ , ( $j = 1, 2, 3$ ). The sum of the expressions obtained for each  $f_j$  must satisfy the boundary conditions of the problem given by (4)-(8).

## 2.1 The contact problem ( $f_1 = 0$ , $f_2 = 0$ )

In this section, we will determine the stress and displacement field due to stamp loading, namely  $f_3(y)$ . This can be accomplished by using Fourier transforms. The displacement components can be expressed as

$$u_3(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} U_3(x, \rho) \exp(i\rho y) d\rho, \quad (10a)$$

$$v_3(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} V_3(x, \rho) \exp(i\rho y) d\rho. \quad (10b)$$

In (10) subscript 3 stands for the displacements due to stamp loading. Substituting (10) in (3) following ordinary differential equations are obtained:

$$(\kappa + 1) \frac{d^2 U_3}{dx^2} + \gamma(\kappa + 1) \frac{dU_3}{dx} - \rho^2(\kappa - 1)U_3 + 2i\rho \frac{dV_3}{dx} + \gamma i\rho(3 - \kappa)V_3 = 0, \quad (11a)$$

$$2i\rho \frac{dU_3}{dx} + \gamma i\rho(\kappa - 1)U_3 + (\kappa - 1) \frac{d^2 V_3}{dx^2} + \gamma(\kappa - 1) \frac{dV_3}{dx} - \rho^2(\kappa + 1)V_3 = 0. \quad (11b)$$

Assuming a solution of the form  $\exp(sx)$  following characteristic equation is obtained,

$$(s^2 + \gamma s - \rho^2 - i|\rho|\delta_3)(s^2 + \gamma s - \rho^2 + i|\rho|\delta_3) = 0, \quad \delta_3 = |\gamma| \sqrt{\frac{3 - \kappa}{\kappa + 1}}. \quad (12a,b)$$

Roots of the characteristic equation are given by

$$s_1 = -\frac{1}{2}\gamma - \frac{1}{2}\sqrt{\gamma^2 + 4\rho^2 + 4i|\rho|\delta_3}, \quad \Re(s_1) < 0, \quad (13a)$$

$$s_2 = -\frac{1}{2}\gamma - \frac{1}{2}\sqrt{\gamma^2 + 4\rho^2 - 4i|\rho|\delta_3}, \quad \Re(s_2) < 0, \quad (13b)$$

$$s_3 = -\frac{1}{2}\gamma + \frac{1}{2}\sqrt{\gamma^2 + 4\rho^2 + 4i|\rho|\delta_3}, \quad \Re(s_3) > 0, \quad (13c)$$

$$s_4 = -\frac{1}{2}\gamma + \frac{1}{2}\sqrt{\gamma^2 + 4\rho^2 - 4i|\rho|\delta_3}, \quad \Re(s_4) > 0. \quad (13d)$$

The displacement components  $u_3$  and  $v_3$  then can be written as

$$u_3(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^2 M_j \exp(s_j x + i\rho y) d\rho, \quad (14a)$$

$$v_3(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^2 M_j N_j \exp(s_j x + i\rho y) d\rho, \quad (14b)$$

where

$$N_j(\rho) = \frac{i((\kappa + 1)s_j^2 + \gamma(\kappa + 1)s_j + \rho^2(1 - \kappa))}{\rho(2s_j + \gamma(3 - \kappa))}, \quad (j = 1, 2, 3, 4). \quad (15)$$

Using (2), stresses and displacement derivative can be expressed as follows:

$$\sigma_{xx3}(x, y) = \frac{\mu(x)}{\kappa - 1} \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^2 (s_j(\kappa + 1) + i\rho N_j(3 - \kappa)) M_j \exp(s_j x + i\rho y) d\rho, \quad (16a)$$

$$\sigma_{yy3}(x, y) = \frac{\mu(x)}{\kappa - 1} \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^2 (s_j(3 - \kappa) + i\rho N_j(\kappa + 1)) M_j \exp(s_j x + i\rho y) d\rho, \quad (16b)$$

$$\sigma_{xy3}(x, y) = \mu(x) \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^2 (i\rho + N_j s_j) M_j \exp(s_j x + i\rho y) d\rho, \quad (16c)$$

$$\frac{\partial}{\partial y} u_3(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} i\rho \sum_{j=1}^2 M_j \exp(s_j x + i\rho y) d\rho. \quad (16d)$$

Using the boundary conditions given by (4) and (5a), we can write,

$$\begin{aligned} \frac{\mu_0}{\kappa - 1} \int_{-\infty}^{\infty} \sum_{j=1}^2 (s_j(\kappa + 1) + i\rho N_j(3 - \kappa)) M_j \exp(i\rho y) d\rho = \\ = \begin{cases} f_3(y), & a < y < b \\ 0, & -\infty < y < a, b < y < \infty \end{cases} \end{aligned} \quad (17a)$$

$$\mu_0 \int_{-\infty}^{\infty} \sum_{j=1}^2 (i\rho + N_j s_j) M_j \exp(i\rho y) d\rho = \begin{cases} \eta f_3(y), & a < y < b \\ 0, & -\infty < y < a, b < y < \infty \end{cases} \quad (17b)$$

We express  $M_j(\rho)$ , ( $j = 1, 2$ ) in the following form,

$$M_j(\rho) = \frac{1}{\mu_0} \psi_j(\rho) \int_a^b f_3(t) \exp(-i\rho t) dt. \quad (18)$$

Then,  $\psi_j(\rho)$ , ( $j = 1, 2$ ) is determined from

$$\sum_{j=1}^2 (s_j(\kappa + 1) + i\rho N_j(3 - \kappa)) \psi_j(\rho) = (\kappa - 1), \quad (19a)$$

$$\sum_{j=1}^2 (i\rho + N_j s_j) \psi_j(\rho) = \eta. \quad (19b)$$

Stresses and displacement derivative for the stamp loading can now be obtained using (16), (18) and (19).

## 2.2 The opening mode problem ( $f_2 = 0, f_3 = 0$ )

In this section we will determine the stresses and displacement derivatives due to relative displacement derivative of the crack faces in  $y$  - direction, namely  $f_1(x)$ . First, we will derive the expressions for stresses and displacement derivative for a crack in infinite plane. Then, the solution for the half-plane ( $x > 0$ ), will be superimposed to satisfy the boundary conditions at the free surface  $x = 0$ . In the solution of the half-plane problem, we will also consider the symmetry about  $x$ -axis. For a crack in infinite plane, displacement components can be expressed using Fourier transformations as follows:

$$u_1^{(i)}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} U_1^{(i)}(\omega, y) \exp(i\omega x) d\omega, \quad (20a)$$

$$v_1^{(i)}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} V_1^{(i)}(\omega, y) \exp(i\omega x) d\omega, \quad (20b)$$

where subscript 1 stands for the opening mode problem (i.e.,  $f_1(x) \neq 0, f_2(x) = 0, f_3(x) = 0$ ) and superscript  $(i)$  stands for the infinite plane problem. Substituting (20) in (11) following differential equations are obtained:

$$(\kappa - 1) \frac{d^2 U_1}{dy^2} + (\kappa + 1)(\gamma i\omega - \omega^2) U_1 + (2i\omega + \gamma(3 - \kappa)) \frac{dV_1}{dy} = 0, \quad (21a)$$

$$(2i\omega + \gamma(\kappa - 1)) \frac{dU_1}{dy} + (\kappa + 1) \frac{d^2 V_1}{dy^2} + (\kappa - 1)(\gamma i\omega - \omega^2) V_1 = 0. \quad (21b)$$

Assuming a solution of the form  $\exp(ny)$  the characteristic equation is found to be

$$(n^2 - \delta_1 n + i\omega(\gamma + i\omega))(n^2 + \delta_1 n + i\omega(\gamma + i\omega)) = 0, \quad \delta_1 = \gamma \sqrt{\frac{3 - \kappa}{\kappa + 1}}. \quad (22a,b)$$

Roots of the characteristic equation are given by

$$n_1 = -\frac{1}{2}\delta_1 + \frac{1}{2}\sqrt{4\omega^2 - 4i\omega\gamma + \delta_1^2}, \quad \Re(n_1) > 0, \quad (23a)$$

$$n_2 = \frac{1}{2}\delta_1 + \frac{1}{2}\sqrt{4\omega^2 - 4i\omega\gamma + \delta_1^2}, \quad \Re(n_2) > 0, \quad (23b)$$

$$n_3 = -\frac{1}{2}\delta_1 - \frac{1}{2}\sqrt{4\omega^2 - 4i\omega\gamma + \delta_1^2}, \quad \Re(n_3) < 0, \quad (23c)$$

$$n_4 = \frac{1}{2}\delta_1 - \frac{1}{2}\sqrt{4\omega^2 - 4i\omega\gamma + \delta_1^2}, \quad \Re(n_4) < 0. \quad (23d)$$

Then, for  $y < 0$  and  $y > 0$  stresses and displacements can be expressed as follows:

$$y < 0$$

$$u_1^{(i^-)}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^2 C_j \exp(n_j y + i\omega x) d\omega, \quad (24a)$$

$$v_1^{(i^-)}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^2 C_j A_j \exp(n_j y + i\omega x) d\omega, \quad (24b)$$

$$\sigma_{xx1}^{(i^-)}(x, y) = \frac{\mu(x)}{\kappa - 1} \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^2 S_{xx1j}^{(i^-)}(\omega) \exp(n_j y + i\omega x) d\omega, \quad (24c)$$

$$\sigma_{yy1}^{(i^-)}(x, y) = \frac{\mu(x)}{\kappa - 1} \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^2 S_{yy1j}^{(i^-)}(\omega) \exp(n_j y + i\omega x) d\omega, \quad (24d)$$

$$\sigma_{xy1}^{(i^-)}(x, y) = \mu(x) \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^2 (n_j + i\omega A_j) C_j \exp(n_j y + i\omega x) d\omega, \quad (24e)$$

$$\frac{\partial}{\partial y} u_1^{(i^-)}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^2 C_j n_j \exp(n_j y + i\omega x) d\omega, \quad (24f)$$

$$S_{xx1j}^{(i^-)} = \sum_{j=1}^2 (i\omega(\kappa + 1) + A_j n_j(3 - \kappa)) C_j, \quad (24g)$$

$$S_{yy1j}^{(i^-)} = \sum_{j=1}^2 (i\omega(3 - \kappa) + A_j n_j(\kappa + 1)) C_j, \quad (24h)$$

where superscript  $(i^-)$  stands for  $y < 0$ .

$$y > 0$$

$$u_1^{(i^+)}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=3}^4 C_j \exp(n_j y + i\omega x) d\omega, \quad (25a)$$

$$v_1^{(i^+)}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=3}^4 C_j A_j \exp(n_j y + i\omega x) d\omega, \quad (25b)$$

$$\sigma_{xx1}^{(i^+)}(x, y) = \frac{\mu(x)}{\kappa - 1} \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=3}^4 S_{xx1j}^{(i^+)}(\omega) \exp(n_j y + i\omega x) d\omega, \quad (25c)$$

$$\sigma_{yy1}^{(i^+)}(x, y) = \frac{\mu(x)}{\kappa - 1} \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=3}^4 S_{yy1j}^{(i^+)}(\omega) \exp(n_j y + i\omega x) d\omega, \quad (25d)$$

$$\sigma_{xy1}^{(i^+)}(x, y) = \mu(x) \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=3}^4 (n_j + i\omega A_j) C_j \exp(n_j y + i\omega x) d\omega, \quad (25e)$$

$$\frac{\partial}{\partial y} u_1^{(i^+)}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=3}^4 C_j n_j \exp(n_j y + i\omega x) d\omega, \quad (25f)$$

$$S_{xx1j}^{(i^+)} = \sum_{j=3}^4 (i\omega(\kappa + 1) + A_j n_j (3 - \kappa)) C_j, \quad (25g)$$

$$S_{yy1j}^{(i^+)} = \sum_{j=3}^4 (i\omega(3 - \kappa) + A_j n_j (\kappa + 1)) C_j, \quad (25h)$$

where superscript  $(i^+)$  stands for  $y > 0$ . In (24) and (25)  $C_j$  ( $j = 1, 2, 3, 4$ ) are unknown constants and  $A_j$  is given by

$$A_j(\omega) = - \frac{n_j^2(\kappa - 1) + (i\omega\gamma - \omega^2)(\kappa + 1)}{n_j(2i\omega + \gamma(3 - \kappa))}. \quad (26)$$

For a crack in infinite plane and for the opening mode following boundary conditions must be satisfied:

$$\sigma_{yy1}^{(i^+)}(x, 0) = \sigma_{yy1}^{(i^-)}(x, 0), \quad -\infty < x < \infty, \quad (27a)$$

$$\sigma_{xy1}^{(i^+)}(x, 0) = \sigma_{xy1}^{(i^-)}(x, 0), \quad -\infty < x < \infty, \quad (27b)$$

$$\frac{2\mu_0}{\kappa+1} \frac{\partial}{\partial x} \left( u_1^{(i^+)}(x, 0) - u_1^{(i^-)}(x, 0) \right) = 0, \quad -\infty < x < \infty \quad (27c)$$

$$\frac{2\mu_0}{\kappa+1} \frac{\partial}{\partial x} \left( v_1^{(i^+)}(x, 0) - v_1^{(i^-)}(x, 0) \right) = \begin{cases} 0, & -\infty < x < 0, d < x < \infty \\ f_1(x), & 0 < x < d \end{cases} \quad (27d)$$

We first express the unknown constants  $C_j(\omega)$  ( $j = 1, 2, 3, 4$ ) as

$$C_j(\omega) = \frac{\kappa+1}{2\mu_0} P_j(\omega) \int_0^d f_1(t) \exp(-i\omega t) dt. \quad (28)$$

Then, by using (24), (25) and (27) the following equations can be obtained to determine

$$P_j(\omega)$$

$$\sum_{j=3}^4 (i\omega(3-\kappa) + A_j n_j(1+\kappa)) P_j(\omega) - \sum_{j=1}^2 (i\omega(3-\kappa) + A_j n_j(1+\kappa)) P_j(\omega) = 0, \quad (29a)$$

$$\sum_{j=3}^4 (n_j + i\omega A_j) P_j(\omega) - \sum_{j=1}^2 (n_j + i\omega A_j) P_j(\omega) = 0, \quad (29b)$$

$$i\omega \left\{ \sum_{j=3}^4 A_j P_j(\omega) - \sum_{j=1}^2 A_j P_j(\omega) \right\} = 1, \quad (29c)$$

$$i\omega \left\{ P_4(\omega) + P_3(\omega) - P_2(\omega) - P_1(\omega) \right\} = 0. \quad (29d)$$

Using (34), (25) and (29) the stresses and the displacement derivative for the opening mode can be obtained. Note that, if we solve (29) for  $P_j(\omega)$  then substitute in (28) and (24e), (25e), we find that

$$\sigma_{xy1}^{(i^+)}(x, 0) = \sigma_{xy1}^{(i^-)}(x, 0) = 0 \quad (30)$$

which is expected due to the symmetry about  $x$ -axis. For the opening mode problem  $u_1(x, y)$  is an even and  $v_1(x, y)$  is an odd function of  $y$ . In order to satisfy the free surface boundary conditions at  $x = 0$ , we will now superimpose the solution for the half-plane ( $x > 0$ ) on the solution for infinite plane, and because of symmetry in the half-plane ( $x > 0$ ) solution we will only consider  $y > 0$ . Hence, displacement components for the half-plane can be written by using the following Fourier cosine and sine integrals:

$$u_1^{(h)}(x, y) = \int_0^\infty U_1^{(h)}(x, \alpha) \cos(\alpha y) d\alpha, \quad (31a)$$

$$v_1^{(h)}(x, y) = \int_0^\infty V_1^{(h)}(x, \alpha) \sin(\alpha y) d\alpha, \quad (31b)$$

where superscript  $(h)$  stands for the half-plane problem. Substituting (31) in (3) following ordinary differential equations are obtained:

$$(\kappa + 1) \frac{d^2 U_1^{(h)}}{dx^2} + \gamma(\kappa - 1) \frac{dU_1^{(h)}}{dx} - \alpha^2(\kappa - 1) U_1^{(h)} + 2\alpha \frac{dV_1^{(h)}}{dx} + \gamma(3 - \kappa) V_1^{(h)} = 0, \quad (32a)$$

$$- 2\alpha \frac{dU_1^{(h)}}{dx} - \gamma\alpha(\kappa - 1) U_1^{(h)} + (\kappa - 1) \frac{d^2 V_1^{(h)}}{dx^2} + \gamma(\kappa - 1) \frac{dV_1^{(h)}}{dx} - \alpha^2(\kappa + 1) V_1^{(h)} = 0. \quad (32b)$$

Assuming a solution of the form  $\exp(px)$  we obtain the characteristic equation as,

$$(p^2 + \gamma p - \alpha^2 - i\alpha\delta_1)(p^2 + \gamma p - \alpha^2 + i\alpha\delta_1) = 0, \quad (33)$$

where  $\delta_1$  is given by (22b). Roots of the characteristic equation are found to be

$$p_1 = -\frac{1}{2}\gamma + \frac{1}{2}\sqrt{\gamma^2 + 4\alpha^2 + 4i\alpha\delta_1}, \quad \Re(p_1) > 0, \quad (34a)$$

$$p_2 = -\frac{1}{2}\gamma + \frac{1}{2}\sqrt{\gamma^2 + 4\alpha^2 - 4i\alpha\delta_1}, \quad \Re(p_2) > 0, \quad (34b)$$

$$p_3 = -\frac{1}{2}\gamma - \frac{1}{2}\sqrt{\gamma^2 + 4\alpha^2 + 4i\alpha\delta_1}, \quad \Re(p_3) < 0, \quad (34c)$$

$$p_4 = -\frac{1}{2}\gamma - \frac{1}{2}\sqrt{\gamma^2 + 4\alpha^2 - 4i\alpha\delta_1}, \quad \Re(p_4) < 0. \quad (34d)$$

The stresses and displacements for the half-plane problem are then expressed as follows:

$$u_1^{(h)}(x, y) = \int_0^\infty (B_3 \exp(p_3 x) + B_4 \exp(p_4 x)) \cos(\alpha y) d\alpha, \quad (35a)$$

$$v_1^{(h)}(x, y) = \int_0^\infty (B_3 D_3 \exp(p_3 x) + B_4 D_4 \exp(p_4 x)) \sin(\alpha y) d\alpha, \quad (35b)$$

$$\sigma_{xx1}^{(h)}(x, y) = \frac{\mu(x)}{\kappa - 1} \int_0^\infty \sum_{j=3}^4 ((\kappa + 1)p_j + D_j \alpha(3 - \kappa)) B_j \exp(p_j x) \cos(\alpha y) d\alpha, \quad (35c)$$

$$\sigma_{yy1}^{(h)}(x, y) = \frac{\mu(x)}{\kappa - 1} \int_0^\infty \sum_{j=3}^4 ((3 - \kappa)p_j + D_j \alpha(\kappa + 1)) B_j \exp(p_j x) \cos(\alpha y) d\alpha, \quad (35d)$$

$$\sigma_{xy1}^{(h)}(x, y) = \mu(x) \int_0^\infty \sum_{j=3}^4 (D_j p_j - \alpha) B_j \exp(p_j x) \sin(\alpha y) d\alpha, \quad (35e)$$

$$\frac{\partial}{\partial y} u_1^{(h)}(x, y) = - \int_0^\infty \alpha (B_3 \exp(p_3 x) + B_4 \exp(p_4 x)) \sin(\alpha y) d\alpha, \quad (35f)$$

where  $B_j$  ( $j = 3, 4$ ) are unknown constants and  $D_j$  is given by,

$$D_j = - \frac{p_j^2(\kappa + 1) + \alpha^2(1 - \kappa) + \gamma p_j(1 + \kappa)}{\alpha(2p_j + \gamma(3 - \kappa))}. \quad (36)$$

For  $x > 0$  and  $y > 0$ , the total stress and displacement fields can be obtained by adding the equations (25) and (35), that is

$$u_1(x, y) = u_1^{(i^+)}(x, y) + u_1^{(h)}(x, y), \quad (37a)$$

$$v_1(x, y) = v_1^{(i^+)}(x, y) + v_1^{(h)}(x, y), \quad (37b)$$

$$\sigma_{xx1}(x, y) = \sigma_{xx1}^{(i^+)}(x, y) + \sigma_{xx1}^{(h)}(x, y), \quad (37c)$$

$$\sigma_{xy1}(x, y) = \sigma_{xy1}^{(i^+)}(x, y) + \sigma_{xy1}^{(h)}(x, y), \quad (37d)$$

$$\sigma_{yy1}(x, y) = \sigma_{yy1}^{(i^+)}(x, y) + \sigma_{yy1}^{(h)}(x, y). \quad (37e)$$

The constants  $B_j(\alpha)$ , ( $j = 3, 4$ ) are determined by using the free surface boundary conditions as follows,

$$\sigma_{xx1}(0, y) = \sigma_{xx1}^{(i^+)}(0, y) + \sigma_{xx1}^{(h)}(0, y) = 0, \quad 0 < y < \infty, \quad (38a)$$

$$\sigma_{xy1}(0, y) = \sigma_{xy1}^{(i^+)}(0, y) + \sigma_{xy1}^{(h)}(0, y) = 0, \quad 0 < y < \infty. \quad (38b)$$

Note that due to symmetry we only consider  $0 < y < \infty$ . Using (25c), (25e), (35c), and (35e) and after simplifications using by MAPLE, (38) is reduced to following form:

$$\begin{aligned} \sum_{j=3}^4 ((\kappa + 1)p_j + D_j\alpha(3 - \kappa))B_j(\alpha) &= \\ &= -\frac{1}{\pi^2} \frac{\kappa - 1}{2\mu_0} \int_0^d f_1(t) dt \int_{-\infty}^{\infty} F_{xx1}(\omega, \alpha) \exp(-i\omega t) d\omega, \end{aligned} \quad (39a)$$

$$\sum_{j=3}^4 (D_j p_j - \alpha) B_j(\alpha) = -\frac{1}{\pi^2} \frac{1}{2\mu_0} \int_0^d f_1(t) dt \int_{-\infty}^{\infty} F_{yy1}(\omega, \alpha) \exp(-i\omega t) d\omega \quad (39b)$$

where,

$$F_{xx1}(\omega, \alpha) = \frac{4i\alpha^2\omega}{D(\omega, \alpha)}, \quad F_{yy1}(\omega, \alpha) = \frac{4\omega\alpha(\omega - i\gamma)}{D(\omega, \alpha)}, \quad (40a,b)$$

$$D(\omega, \alpha) = \omega^4 - 2i\gamma\omega^3 + (2\alpha^2 - \gamma^2)\omega^2 - 2i\alpha^2\gamma\omega + \alpha^4 + \alpha^2\gamma^2 \frac{3 - \kappa}{\kappa + 1}. \quad (40c)$$

The inner integrals in (39) are evaluated in closed form using the theory of residues and (39) is reduced to following form to determine the unknown constants  $B_j(\alpha)$ , ( $j = 3, 4$ ):

$$\sum_{j=3}^4 ((\kappa + 1)p_j + D_j\alpha(3 - \kappa))B_j^*(\alpha, t) = R_{xx1}(\alpha, t), \quad (41a)$$

$$\sum_{j=3}^4 (D_j p_j - \alpha) B_j^*(\alpha, t) = R_{xy1}(\alpha, t). \quad (41b)$$

$B_j(\alpha)$  is now defined as

$$B_j(\alpha) = \frac{\kappa+1}{2\mu_0} \int_0^d B_j^*(\alpha, t) \exp\left(\left(\frac{\gamma}{2} - \lambda_1\right)t\right) f_1(t) dt, \quad (42a)$$

and,

$$R_{xx1}(\alpha, t) = \frac{1}{\pi} \frac{\kappa-1}{\kappa+1} \frac{\alpha^2}{\lambda_1 \lambda_2 (\lambda_1^2 + \lambda_2^2)} \left\{ \gamma \lambda_2 \cos(\lambda_2 t) + (\gamma \lambda_1 - 2(\lambda_1^2 + \lambda_2^2) \sin(\lambda_2 t)) \right\} \quad (42b)$$

$$R_{xy1}(\alpha, t) = -\frac{2}{\pi} \frac{1}{\kappa+1} \frac{\alpha}{\lambda_1 \lambda_2 (\lambda_1^2 + \lambda_2^2)} \times \left\{ \lambda_2 (\lambda_1^2 + \lambda_2^2 + \gamma^2/4) \cos(\lambda_2 t) - \lambda_1 (\lambda_1^2 + \lambda_2^2 - \gamma^2/4) \sin(\lambda_2 t) \right\} \quad (42c)$$

where

$$\lambda_1 = \sqrt{\frac{R_1 + R_2}{2}}, \quad \lambda_2 = \sqrt{\frac{R_1 - R_2}{2}}, \quad (43a,b)$$

$$R_1 = \sqrt{(\gamma^2/4 + \alpha^2)^2 + \alpha^2 \gamma^2 (3 - \kappa) / (\kappa + 1)}, \quad (43c)$$

$$R_2 = \gamma^2/4 + \alpha^2. \quad (43d)$$

This completes the formulation for the opening mode problem. Stresses and displacements are given by (37) and unknown constants are given by (29) and (41).

### 2.3 The sliding mode problem ( $f_1 = 0, f_3 = 0$ )

In this section, we will determine the displacements and stresses due to relative displacement derivative of the crack faces in  $x$ -direction, namely  $f_2(x)$ . First we will

derive the expressions for stresses and displacements for a crack in infinite plane. Then, the solution for the half-plane ( $x > 0$ ) will be superimposed to satisfy the boundary conditions at the free surface  $x = 0$ . Again, symmetry will be considered in the solution of the half-plane ( $x > 0$ ) problem. Following a similar procedure as given in Section 2.2, stresses and displacements for the infinite plane can be written as follows,

$$y < 0$$

$$u_2^{(i^-)}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^2 E_j \exp(n_j y + i\omega x) d\omega, \quad (44a)$$

$$v_2^{(i^-)}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^2 E_j A_j \exp(n_j y + i\omega x) d\omega, \quad (44b)$$

$$\sigma_{xx2}^{(i^-)}(x, y) = \frac{\mu(x)}{\kappa - 1} \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^2 S_{xx2j}^{(i^-)}(\omega) \exp(n_j y + i\omega x) d\omega, \quad (44c)$$

$$\sigma_{yy2}^{(i^-)}(x, y) = \frac{\mu(x)}{\kappa - 1} \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^2 S_{yy2j}^{(i^-)}(\omega) \exp(n_j y + i\omega x) d\omega, \quad (44d)$$

$$\sigma_{xy2}^{(i^-)}(x, y) = \mu(x) \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^2 (n_j + i\omega A_j) E_j \exp(n_j y + i\omega x) d\omega, \quad (44e)$$

$$\frac{\partial}{\partial y} u_2^{(i^-)}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=1}^2 E_j n_j \exp(n_j y + i\omega x) d\omega, \quad (44f)$$

$$S_{xx2j}^{(i^-)} = \sum_{j=1}^2 (i\omega(\kappa + 1) + A_j n_j(3 - \kappa)) E_j, \quad (44g)$$

$$S_{yy2j}^{(i^-)} = \sum_{j=1}^2 (i\omega(3 - \kappa) + A_j n_j(\kappa + 1)) E_j, \quad (44h)$$

where subscript 2 stands for the sliding mode or mode II problem and superscript  $(i^-)$  stands for  $y < 0$ .

$$y > 0$$

$$u_2^{(i^+)}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=3}^4 E_j \exp(n_j y + i\omega x) d\omega, \quad (45a)$$

$$v_2^{(i^+)}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=3}^4 E_j A_j \exp(n_j y + i\omega x) d\omega, \quad (45b)$$

$$\sigma_{xx2}^{(i^+)}(x, y) = \frac{\mu(x)}{\kappa - 1} \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=3}^4 S_{xx2j}^{(i^+)}(\omega) \exp(n_j y + i\omega x) d\omega, \quad (45c)$$

$$\sigma_{yy2}^{(i^+)}(x, y) = \frac{\mu(x)}{\kappa - 1} \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=3}^4 S_{yy2j}^{(i^+)}(\omega) \exp(n_j y + i\omega x) d\omega, \quad (45d)$$

$$\sigma_{xy2}^{(i^+)}(x, y) = \mu(x) \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=3}^4 (n_j + i\omega A_j) E_j \exp(n_j y + i\omega x) d\omega, \quad (45e)$$

$$\frac{\partial}{\partial y} u_2^{(i^+)}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=3}^4 E_j n_j \exp(n_j y + i\omega x) d\omega, \quad (45f)$$

$$S_{xx2j}^{(i^+)} = \sum_{j=3}^4 (i\omega(\kappa + 1) + A_j n_j (3 - \kappa)) E_j, \quad (45g)$$

$$S_{yy2j}^{(i^+)} = \sum_{j=3}^4 (i\omega(3 - \kappa) + A_j n_j (\kappa + 1)) E_j, \quad (45h)$$

where superscript  $(i^+)$  stands for the half-plane  $(y > 0)$ ,  $n_j$ ,  $(j = 1, 2, 3, 4)$  is given by (23) and  $A_j$  is given by (26). The unknown constants  $E_j$ ,  $(j = 1, 2, 3, 4)$  are determined using the following boundary conditions:

$$\sigma_{yy2}^{(i^+)}(x, 0) = \sigma_{yy2}^{(i^-)}(x, 0), \quad -\infty < x < \infty, \quad (46a)$$

$$\sigma_{xy2}^{(i^+)}(x, 0) = \sigma_{xy2}^{(i^-)}(x, 0), \quad -\infty < x < \infty, \quad (46b)$$

$$\frac{2\mu_0}{\kappa + 1} \frac{\partial}{\partial x} \left( u_2^{(i^+)}(x, 0) - u_2^{(i^-)}(x, 0) \right) = \begin{cases} 0, & -\infty < x < 0, d < x < \infty \\ f_2(x), & 0 < x < d \end{cases} \quad (46c)$$

$$\frac{2\mu_0}{\kappa+1} \frac{\partial}{\partial x} \left( v_2^{(i^+)}(x, 0) - v_2^{(i^-)}(x, 0) \right) = 0, \quad -\infty < x < \infty. \quad (46d)$$

We first express the unknown constants  $E_j(\omega)$ , in the following form

$$E_j(\omega) = \frac{\kappa+1}{2\mu_0} Z_j(\omega) \int_0^d f_2(t) \exp(-i\omega t) dt. \quad (47)$$

Then, using (44), (45) and (46) following equations are obtained to determine  $Z_j(\omega)$ :

$$\sum_{j=3}^4 (i\omega(3-\kappa) + A_j n_j(1+\kappa)) Z_j(\omega) - \sum_{j=1}^2 (i\omega(3-\kappa) + A_j n_j(1+\kappa)) Z_j(\omega) = 0, \quad (48a)$$

$$\sum_{j=3}^4 (n_j + i\omega A_j) Z_j(\omega) - \sum_{j=1}^2 (n_j + i\omega A_j) Z_j(\omega) = 0, \quad (48b)$$

$$i\omega \left\{ \sum_{j=3}^4 A_j Z_j(\omega) - \sum_{j=1}^2 A_j Z_j(\omega) \right\} = 0, \quad (48c)$$

$$i\omega \left\{ Z_4(\omega) + Z_3(\omega) - Z_2(\omega) - Z_1(\omega) \right\} = 1. \quad (48d)$$

By using (44), (45), (47) and (48) stresses and displacement derivative for the sliding mode problem can then be obtained. Note that if we solve (48) for  $Z_j(\omega)$  and substitute the results in (47), (44d) and (45d) we find that,

$$\sigma_{yy2}^{(i^+)}(x, 0) = \sigma_{yy2}^{(i^-)}(x, 0) = 0, \quad (49)$$

which is expected due to the symmetry about the  $x$ -axis. For the sliding mode problem  $u_2(x, y)$  is an odd and  $v_2(x, y)$  is an even function of  $y$ . In order to satisfy the free surface boundary conditions at  $x = 0$ , we will now superimpose the solution for the half-plane ( $x > 0$ ) on the solution for the infinite plane, and because of symmetry, in the half-plane ( $x > 0$ ) solution we will only consider  $y > 0$ . Hence, displacement components for the half-plane can be written using Fourier sine and cosine integrals as follows:

$$u_2^{(h)}(x, y) = \int_0^\infty U_2^{(h)}(x, \alpha) \sin(\alpha y) d\alpha, \quad (50a)$$

$$v_2^{(h)}(x, y) = \int_0^\infty V_2^{(h)}(x, \alpha) \cos(\alpha y) d\alpha, \quad (50b)$$

where superscript  $(h)$  stands for the half-plane problem. Substituting (50) in (3) following ordinary differential equations are obtained

$$(\kappa + 1) \frac{d^2 U_2^{(h)}}{dx^2} + \gamma(\kappa + 1) \frac{dU_2^{(h)}}{dx} - \alpha^2(\kappa - 1) U_2^{(h)} - 2\alpha \frac{dV_2^{(h)}}{dx} - \gamma\alpha(3 - \kappa) V_2^{(h)} = 0, \quad (51a)$$

$$2\alpha \frac{dU_2^{(h)}}{dx} + \gamma\alpha(\kappa - 1) U_2^{(h)} + (\kappa - 1) \frac{d^2 V_2^{(h)}}{dx^2} + \gamma(\kappa - 1) \frac{dV_2^{(h)}}{dx} - \alpha^2(\kappa + 1) V_2^{(h)} = 0. \quad (51b)$$

Assuming a solution of the form  $\exp(tx)$ , the characteristic equation is obtained as

$$(t^2 + \gamma t - \alpha^2 - i\alpha\delta_1)(t^2 + \gamma t - \alpha^2 + i\alpha\delta_1) = 0, \quad (52)$$

where  $\delta_1$  is given by (22b). Roots of the characteristic equation are obtained as follows

$$t_1 = -\frac{1}{2}\gamma + \frac{1}{2}\sqrt{\gamma^2 + 4\alpha^2 + 4i\alpha\delta_1}, \quad \Re(t_1) > 0, \quad (53a)$$

$$t_2 = -\frac{1}{2}\gamma + \frac{1}{2}\sqrt{\gamma^2 + 4\alpha^2 - 4i\alpha\delta_1}, \quad \Re(t_2) > 0, \quad (53b)$$

$$t_3 = -\frac{1}{2}\gamma - \frac{1}{2}\sqrt{\gamma^2 + 4\alpha^2 + 4i\alpha\delta_1}, \quad \Re(t_3) < 0, \quad (53c)$$

$$t_4 = -\frac{1}{2}\gamma - \frac{1}{2}\sqrt{\gamma^2 + 4\alpha^2 - 4i\alpha\delta_1}, \quad \Re(t_4) < 0. \quad (53d)$$

For the half-plane solution the stresses and displacements can now be expressed in the following form

$$u_2^{(h)}(x, y) = \int_0^\infty (G_3 \exp(t_3 x) + G_4 \exp(t_4 x)) \sin(\alpha y) d\alpha, \quad (54a)$$

$$v_2^{(h)}(x, y) = \int_0^\infty (G_3 H_3 \exp(t_3 x) + G_4 H_4 \exp(t_4 x)) d\alpha, \quad (54b)$$

$$\sigma_{xx2}^{(h)}(x, y) = \frac{\mu(x)}{\kappa - 1} \int_0^\infty \sum_{j=3}^4 ((\kappa + 1)t_j - \alpha H_j(3 - \kappa)) G_j \exp(t_j x) \sin(\alpha y) d\alpha, \quad (54c)$$

$$\sigma_{yy2}^{(h)}(x, y) = \frac{\mu(x)}{\kappa - 1} \int_0^\infty \sum_{j=3}^4 ((3 - \kappa)t_j - \alpha H_j(\kappa + 1)) G_j \exp(t_j x) \sin(\alpha y) d\alpha, \quad (54d)$$

$$\sigma_{xy2}^{(h)}(x, y) = \mu(x) \int_0^\infty \sum_{j=3}^4 (\alpha + H_j t_j) G_j \exp(t_j x) \cos(\alpha y) d\alpha, \quad (54e)$$

$$\frac{\partial}{\partial y} u_2^{(h)}(x, y) = \int_0^\infty \alpha (G_3 \exp(t_3 x) + G_4 \exp(t_4 x)) \cos(\alpha y) d\alpha, \quad (54f)$$

where  $G_j$ , ( $j = 3, 4$ ) are unknown constants and  $H_j$  is given by,

$$H_j(\alpha) = \frac{\gamma t_j(\kappa + 1) + \alpha^2(1 - \kappa) + t_j^2(\kappa + 1)}{\alpha(2t_j + \gamma(3 - \kappa))}, \quad (j = 3, 4). \quad (55)$$

For  $x > 0$  and  $y > 0$ , the total stress and displacement fields can now be obtained by adding (45) and (54) as

$$u_2(x, y) = u_2^{(i^+)}(x, y) + u_2^{(h)}(x, y), \quad (56a)$$

$$v_2(x, y) = v_2^{(i^+)}(x, y) + v_2^{(h)}(x, y), \quad (56b)$$

$$\sigma_{xx2}(x, y) = \sigma_{xx2}^{(i^+)}(x, y) + \sigma_{xx2}^{(h)}(x, y), \quad (56c)$$

$$\sigma_{xy2}(x, y) = \sigma_{xy2}^{(i^+)}(x, y) + \sigma_{xy2}^{(h)}(x, y), \quad (56d)$$

$$\sigma_{yy2}(x, y) = \sigma_{yy2}^{(i^+)}(x, y) + \sigma_{yy2}^{(h)}(x, y). \quad (56e)$$

The constants  $G_j(\alpha)$ , ( $j = 3, 4$ ) are determined by using the free surface boundary conditions as follows:

$$\sigma_{xx2}(0, y) = \sigma_{xx2}^{(i^+)}(0, y) + \sigma_{xx2}^{(h)}(0, y) = 0, \quad 0 < y < \infty, \quad (57a)$$

$$\sigma_{xy2}(0, y) = \sigma_{xy2}^{(i^+)}(0, y) + \sigma_{xy2}^{(h)}(0, y) = 0, \quad 0 < y < \infty. \quad (57b)$$

Note that due to symmetry we only consider  $0 < y < \infty$ . Using (45c), (45e), (54c) and (54e) and after simplifications by using MAPLE, (57) is reduced to following form:

$$\begin{aligned} \sum_{j=3}^4 ((\kappa + 1)t_j - H_j \alpha (3 - \kappa)) G_j(\alpha) &= \\ &= -\frac{1}{\pi^2} \frac{\kappa - 1}{2\mu_0} \int_0^d f_2(t) dt \int_{-\infty}^{\infty} F_{xx2}(\omega, \alpha) \exp(-i\omega t) d\omega, \end{aligned} \quad (58a)$$

$$\sum_{j=3}^4 (H_j t_j + \alpha) G_j(\alpha) = -\frac{1}{\pi^2} \frac{1}{2\mu_0} \int_0^d f_2(t) dt \int_{-\infty}^{\infty} F_{xy2}(\omega, \alpha) \exp(-i\omega t) d\omega, \quad (58b)$$

where,

$$F_{xx2}(\omega, \alpha) = \frac{4\alpha^3}{D(\omega, \alpha)}, \quad F_{xy2}(\omega, \alpha) = \frac{4\alpha^2(i\omega + \gamma)}{D(\omega, \alpha)}, \quad (59a,b)$$

and  $D(\omega, \alpha)$  is given by (40c). The inner integrals in (58) are evaluated in closed form by using the theory of residues and (58) is reduced to following form to determine the unknown constants  $G_j(\alpha)$ , ( $j = 3, 4$ ):

$$\sum_{j=3}^4 ((\kappa + 1)t_j - H_j \alpha (3 - \kappa)) G_j^*(\alpha, t) = R_{xx2}(\alpha, t), \quad (60a)$$

$$\sum_{j=3}^4 (H_j t_j + \alpha) G_j^*(\alpha, t) = R_{xy2}(\alpha, t). \quad (60b)$$

where

$$G_j(\alpha) = \frac{\kappa + 1}{2\mu_0} \int_0^d G_j^*(\alpha, t) \exp\left(\left(\frac{\gamma}{2} - \lambda_1\right)t\right) f_2(t) dt \quad (61a)$$

$$R_{xx2}(\alpha, t) = -\frac{2\kappa-1}{\pi\kappa+1} \frac{\alpha^3}{\lambda_1\lambda_2(\lambda_1^2+\lambda_2^2)} \left\{ \lambda_2 \cos(\lambda_2 t) + \lambda_1 \sin(\lambda_2 t) \right\}, \quad (61b)$$

$$R_{xy2}(\alpha, t) = -\frac{1}{\pi\kappa+1} \frac{1}{\lambda_1\lambda_2(\lambda_1^2+\lambda_2^2)} \times \\ \times \left\{ \gamma \lambda_2 \cos(\lambda_2 t) + \left( 2(\lambda_1^2 + \lambda_2^2) + \gamma \lambda_1 \right) \sin(\lambda_2 t) \right\}. \quad (61c)$$

In (61)  $\lambda_1$  and  $\lambda_2$  are given by (43). This completes the formulation for the sliding mode problem. Stresses and displacements are given by (56) and unknown constants are given by (48) and (60).

### 3. Derivation of the singular integral equations

The stress and displacement fields for the contact, opening mode and sliding mode crack problems are given in Section 2 in terms of the unknown functions  $f_1(t)$ ,  $f_2(t)$  and  $f_3(t)$ . The total stress and displacement fields can now be expressed as follows,

$$u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y), \quad 0 < x < \infty, \quad 0 < y < \infty, \quad (62a)$$

$$v(x, y) = v_1(x, y) + v_2(x, y) + v_3(x, y), \quad 0 < x < \infty, \quad 0 < y < \infty, \quad (62b)$$

$$\sigma_{xx}(x, y) = \sigma_{xx1}(x, y) + \sigma_{xx2}(x, y) + \sigma_{xx3}(x, y), \quad 0 < x < \infty, \quad 0 < y < \infty, \quad (62c)$$

$$\sigma_{xy}(x, y) = \sigma_{xy1}(x, y) + \sigma_{xy2}(x, y) + \sigma_{xy3}(x, y), \quad 0 < x < \infty, \quad 0 < y < \infty, \quad (62d)$$

$$\sigma_{yy}(x, y) = \sigma_{yy1}(x, y) + \sigma_{yy2}(x, y) + \sigma_{yy3}(x, y) + E(x)\epsilon_0, \\ 0 < x < \infty, \quad 0 < y < \infty, \quad (62e)$$

where expressions due to  $f_1$ ,  $f_2$  and  $f_3$  are given by (37), (56) and (16) respectively. In this section, the problem will be reduced to three singular integral equations using the boundary conditions (5b) and (6), i.e., by using,

$$\sigma_{yy}(x, 0) = 0, \quad 0 < x < d, \quad (63a)$$

$$\sigma_{xy}(x, 0) = 0, \quad 0 < x < d, \quad (63b)$$

$$\frac{4\mu_0}{\kappa + 1} \frac{\partial}{\partial y} u(0, y) = f(y), \quad a < y < b. \quad (63c)$$

Considering the expressions given in Section 2, stresses and displacement derivative can be written in the following general form,

$$\sigma_{yy}(x, y) = \int_0^d \sum_{j=1}^2 k_{1j}(x, y, t) f_j(t) dt + \int_a^b k_{13}(x, y, t) f_3(t) dt + E(x) \epsilon_0, \quad (64a)$$

$$\sigma_{xy}(x, y) = \int_0^d \sum_{j=1}^2 k_{2j}(x, y, t) f_j(t) dt + \int_a^b k_{23}(x, y, t) f_3(t) dt, \quad (64b)$$

$$\frac{4\mu_0}{1 + \kappa} \frac{\partial}{\partial y} u(x, y) = \int_0^d \sum_{j=1}^2 k_{3j}(x, y, t) f_j(t) dt + \int_a^b k_{33}(x, y, t) f_3(t) dt. \quad (64c)$$

Then, using (63) integral equations can be expressed as follows:

$$\sigma_{yy}(x, 0) = \int_0^d \sum_{j=1}^2 k_{1j}(x, 0, t) f_j(t) dt + \int_a^b k_{13}(x, 0, t) f_3(t) dt + E(x) \epsilon_0 = 0, \\ 0 < x < d, \quad (65a)$$

$$\sigma_{xy}(x, 0) = \int_0^d \sum_{j=1}^2 k_{2j}(x, 0, t) f_j(t) dt + \int_a^b k_{23}(x, 0, t) f_3(t) dt = 0, \\ 0 < x < d, \quad (65b)$$

$$\frac{4\mu_0}{1 + \kappa} \frac{\partial}{\partial y} u(0, y) = \int_0^d \sum_{j=1}^2 k_{3j}(0, y, t) f_j(t) dt + \int_a^b k_{33}(0, y, t) f_3(t) dt = f(y), \\ a < y < b. \quad (65c)$$

In the previous section it was shown that because of symmetry  $\sigma_{yy2}(x, 0) = 0$ , for  $f_1(t) = 0$ ,  $f_2(t) \neq 0$ ,  $f_3(t) = 0$  and  $\sigma_{xy1}(x, 0) = 0$  for  $f_1(t) \neq 0$ ,  $f_2(t) = 0$ ,  $f_3(t) = 0$ , from which it follows that

$$k_{12}(x, 0, t) = 0, \quad k_{21}(x, 0, t) = 0. \quad (66a,b)$$

The expressions for the other kernels  $k_{ij}(x, y, t)$  ( $k_{ij}(x, 0, t)$  for  $i = 1, 2$ , and  $k_{ij}(0, y, t)$  for  $i = 3$ ) will be given in the following sections. The kernels will be expressed in the following general form,

$$k_{ij}(x, 0, t) = \int_0^\infty K_{ij}(x, t, \rho) d\rho, \quad (i = 1, 2, j = 1, 2, 3), \quad (67a)$$

$$k_{ij}(0, y, t) = \int_0^\infty K_{ij}(y, t, \rho) d\rho, \quad (i = 3, j = 1, 2, 3). \quad (67b)$$

With the exception of one case (that being  $k_{33}(0, y, t)$  for  $\gamma < 0$ ) the integrands in (67) are bounded and continuous for  $\rho < \infty$  and integrable at  $\rho = 0$ . The singular nature of the kernels  $k_{ij}$  is therefore determined by examining the asymptotic behavior of the integrands as  $\rho$  tends to infinity. In the following sections we will give the expressions and details of the asymptotic analyses for the kernels.

### 3.1 $k_{11}(x, y, t)$

We first express  $k_{11}(x, y, t)$  as follows

$$k_{11}(x, y, t) = k_{11}^{(i)}(x, y, t) + k_{11}^{(h)}(x, y, t), \quad (68)$$

where  $k_{11}^{(i)}$  is obtained from the infinite plane solution and  $k_{11}^{(h)}$  is obtained from the half-plane ( $x > 0$ ) solution. Referring to (25d),  $k_{11}^{(i)}(x, y, t)$  can be written as

$$k_{11}^{(i)}(x, y, t) = \frac{\kappa + 1}{\kappa - 1} \frac{\exp(\gamma x)}{4\pi} \int_{-\infty}^{\infty} \phi_{11}^{(i)}(\omega, y) \exp(i\omega(x - t)) d\omega, \quad (69a)$$

$$\phi_{11}^{(i)}(\omega, y) = \sum_{j=3}^4 (i\omega(3 - \kappa) + A_j n_j(1 + \kappa)) P_j(\omega) \exp(n_j y), \quad (69b)$$

where  $n_j$ , ( $j = 1, 2, 3, 4$ ),  $A_j$  and  $P_j$  are given by (23), (26) and (29), respectively.

Changing the limits of integration (69a) can be written as,

$$k_{11}^{(i)}(x, y, t) = \frac{\kappa + 1}{\kappa - 1} \frac{\exp(\gamma x)}{4\pi} \int_0^{\infty} \left\{ K_{111}^{(i)}(\omega, y) \cos(\omega(x - t)) + K_{112}^{(i)}(\omega, y) \sin(\omega(x - t)) \right\} d\omega, \quad (70)$$

where,

$$K_{111}^{(i)}(\omega, y) = \phi_{11}^{(i)}(\omega, y) + \phi_{11}^{(i)}(-\omega, y), \quad (71a)$$

$$K_{112}^{(i)}(\omega, y) = i \left( \phi_{11}^{(i)}(\omega, y) - \phi_{11}^{(i)}(-\omega, y) \right). \quad (71b)$$

In order to extract the singular terms we expand  $K_{111}^{(i)}$  and  $K_{112}^{(i)}$  into series as  $\omega \rightarrow \infty$ .

Following asymptotic expansions are obtained by using MAPLE:

$$K_{111}^{(i)\infty}(\omega, y) = \left\{ \frac{f_{11}}{\omega} + \frac{f_{13}}{\omega^3} + \dots + \frac{f_{111}}{\omega^{11}} + O\left(\frac{1}{\omega^{13}}\right) \right\} \exp(-\omega y), \quad (72a)$$

$$K_{112}^{(i)\infty}(\omega, y) = \left\{ f_{20} + \frac{f_{22}}{\omega^2} + \dots + \frac{f_{212}}{\omega^{12}} + O\left(\frac{1}{\omega^{14}}\right) \right\} \exp(-\omega y), \quad (72b)$$

where the leading terms are

$$f_{20} = -\frac{4(\kappa - 1)}{\kappa + 1}, \quad f_{11} = \frac{2(\kappa - 1)\gamma}{\kappa + 1}. \quad (73a,b)$$

Subtracting the asymptotic expansions from the integrands in (70), using integration cutoff points for the infinite integrals, evaluating some of the integrals in closed form and taking the limit as  $y \rightarrow 0$ , after some manipulations  $k_{11}^{(i)}(x, y, t)$  is reduced to

$$k_{11}^{(i)}(x, 0, t) = \exp(\gamma x) \left\{ \frac{1}{\pi} \frac{1}{t-x} + \frac{\kappa+1}{\kappa-1} \frac{1}{4\pi} \left[ J_{111}^{(i)}(x, t) + J_{112}^{(i)}(x, t) - f_{11} \ln \left( A_{111}^{(i)} |t-x| \right) \right] \right\}, \quad (74)$$

where  $A_{111}^{(i)}$  is an integration cutoff point and,

$$\begin{aligned} J_{111}^{(i)}(x, t) &= \int_0^{A_{111}^{(i)}} K_{111}^{(i)}(\omega, 0) \cos(\omega(x-t)) d\omega \\ &+ \int_{A_{111}^{(i)}}^{\infty} \left[ K_{111}^{(i)}(\omega, 0) - K_{111}^{(i)\infty}(\omega, 0) \right] \cos(\omega(x-t)) d\omega \\ &+ \int_{A_{111}^{(i)}}^{\infty} \left[ K_{111}^{(i)\infty}(\omega, 0) - f_{11}/\omega \right] \cos(\omega(x-t)) d\omega \\ &- f_{11} \left\{ \gamma_0 + \int_0^{A_{111}^{(i)}} \frac{\cos(\alpha) - 1}{\alpha} d\alpha \right\}, \end{aligned} \quad (75a)$$

$$\begin{aligned} J_{112}^{(i)}(x, t) &= \int_0^{A_{112}^{(i)}} \left[ K_{112}^{(i)}(\omega, 0) - f_{20} \right] \sin(\omega(x-t)) d\omega \\ &+ \int_{A_{112}^{(i)}}^{\infty} \left[ K_{112}^{(i)}(\omega, 0) - K_{112}^{(i)\infty}(\omega, 0) \right] \sin(\omega(x-t)) d\omega \\ &+ \int_{A_{112}^{(i)}}^{\infty} \left[ K_{112}^{(i)\infty}(\omega, 0) - f_{20} \right] \sin(\omega(x-t)) d\omega, \end{aligned} \quad (75b)$$

where  $A_{112}^{(i)}$  is another integration cutoff point,  $\gamma_0$  is the Euler number, [10]. Second integrals in (75a) and (75b) will be neglected in numerical computation for sufficiently large values of integration cutoff points. Third integrals in (75a) and (75b) are evaluated in closed form. The expressions used in the evaluation of these integrals are given in Appendix A. Referring to (35d) and (42a),  $k_{11}^{(h)}(x, y, t)$  can be written as follows

$$k_{11}^{(h)}(x, y, t) = \frac{\kappa+1}{\kappa-1} \frac{\exp(\gamma x)}{2} \int_0^{\infty} K_{11}^{(h)}(\alpha, t, x) \cos(\alpha y) d\alpha \quad (76)$$

where

$$K_{11}^{(h)}(\alpha, t, x) = \sum_{j=3}^4 \phi_{yyj}^{(h)}(\alpha) B_j^*(\alpha, t) \exp(p_j x + (\gamma/2 - \lambda_1)t), \quad (77a)$$

$$\phi_{yyj}^{(h)}(\alpha) = p_j(3 - \kappa) + D_j \alpha(1 + \kappa), \quad (j = 3, 4). \quad (77b)$$

$p_j$ , ( $j = 1, 2, 3, 4$ ),  $D_j(\alpha)$  and  $B_j^*(\alpha, t)$  are given by (34), (36) and (41) respectively. In order to extract the singular terms we expand  $K_{11}^{(h)}(\alpha, t, x)$  into series as  $\alpha$  tends to infinity as follows:

$$K_{11}^{(h)\infty}(\alpha, t, x) = \left\{ h_2^* \alpha^2 + h_1^* \alpha + h_0 + \frac{h_1}{\alpha} + \frac{h_2}{\alpha^2} + \dots + \frac{h_7}{\alpha^7} + O\left(\frac{1}{\alpha^8}\right) \right\} \times \exp(\gamma(t - x)/2 - (t + x)\alpha), \quad (78)$$

where the coefficients of the expansion are also functions of  $x$  and  $t$ . The leading term is given by,

$$h_2^* = \frac{16}{\pi} \frac{\kappa - 1}{\kappa + 1} \frac{\sin(\delta_1 x/2) \sin(\delta_1 t/2)}{\delta_1^2}. \quad (79)$$

$\delta_1$  is given by (22b). Subtracting  $K_{11}^{(h)\infty}$  from the integrand in (76), evaluating some of the integrals in closed form, using an integration cutoff point  $A_{11}^{(h)}$  for the infinite integral and taking the limit as  $y \rightarrow 0$ , after some manipulations  $k_{11}^{(h)}$  can be written as

$$k_{11}^{(h)}(x, 0, t) = \frac{\kappa + 1}{\kappa - 1} \frac{\exp(\gamma x)}{2} \left\{ r_{11}^{(h)}(x, t) + J_{11}^{(h)}(x, t) - h_1 \exp(\gamma(t - x)/2) \text{Ei}\left(-A_{11}^{(h)}(t + x)\right) \right\}, \quad (80)$$

where  $\text{Ei}()$  is the exponential integral [10] and,

$$r_{11}^{(h)}(x, t) = \left\{ \frac{2h_2^*}{(t + x)^3} + \frac{h_1^*}{(t + x)^2} + \frac{h_0}{t + x} \right\} \exp(\gamma(t - x)/2), \quad (81a)$$

$$\begin{aligned}
J_{11}^{(h)}(x, t) &= \int_0^{A_{11}^{(h)}} K_{11}^{(h)*}(\alpha, t, x) d\alpha \\
&+ \int_{A_{11}^{(h)}}^{\infty} \left[ K_{11}^{(h)}(\alpha, t, x) - K_{11}^{(h)\infty}(\alpha, t, x) \right] d\alpha \\
&+ \int_{A_{11}^{(h)}}^{\infty} I_{11}^{(h)\infty}(\alpha, t, x) d\alpha,
\end{aligned} \tag{81b}$$

$$K_{11}^{(h)*}(\alpha, t, x) = K_{11}^{(h)}(\alpha, t, x) - (h_2^* \alpha^2 + h_1^* \alpha + h_0) \exp(\gamma(t-x)/2 - (t+x)\alpha), \tag{82a}$$

$$I_{11}^{(h)\infty}(\alpha, t, x) = \left\{ \frac{h_2}{\alpha^2} + \frac{h_3}{\alpha^3} + \cdots + \frac{h_7}{\alpha^7} \right\} \exp(\gamma(t-x)/2 - (t+x)\alpha). \tag{82b}$$

Second integral in (81b) will be neglected in numerical computation for a sufficiently large value of  $A_{11}^{(h)}$ . Third integral in (81b) is evaluated in closed form. The expression used in the evaluation of this integral is given in Appendix A.  $k_{11}^{(i)}(x, 0, t)$  and  $k_{11}^{(h)}(x, 0, t)$  are given by equations (74) and (80) respectively. Adding these two equations  $k_{11}(x, 0, t)$  may be expressed as,

$$k_{11}(x, 0, t) = \exp(\gamma x) \left\{ \frac{1}{\pi} \frac{1}{t-x} + h_{11s}(x, t) + h_{11f}(x, t) \right\}, \tag{83}$$

where,

$$h_{11s}(x, t) = \frac{\kappa + 1}{2(\kappa - 1)} r_{11}^{(h)}(x, t), \tag{84a}$$

$$\begin{aligned}
h_{11f}(x, t) &= \frac{\kappa + 1}{\kappa - 1} \frac{1}{4\pi} \left\{ -f_{11} \ln(A_{111}^{(i)} |t-x|) + J_{111}^{(i)}(x, t) + J_{112}^{(i)}(x, t) \right\} + \\
&+ \frac{\kappa + 1}{2(\kappa - 1)} \left\{ J_{11}^{(h)}(x, t) - h_1 \exp(\gamma(t-x)/2) \text{Ei}(-A_{11}^{(h)}(t+x)) \right\},
\end{aligned} \tag{84b}$$

The terms in (84) are given explicitly by (75) and (81). Note that the first term in (83) is the Cauchy singularity associated with a crack in infinite plane, as for the second term we can write,

$$\lim_{(x,t) \rightarrow 0} h_{11s}(x,t) = \frac{1}{\pi} \left( \frac{1}{t+x} + \frac{2t}{(t+x)^2} - \frac{4t^2}{(t+x)^3} \right). \quad (85)$$

This term becomes singular as  $x$  and  $t$  simultaneously approach zero, and is the standard expression found for edge cracks in homogeneous materials. (see for example, equation (23a) in Dag and Erdogan [8]).

### 3.2 $k_{13}(x, y, t)$

Referring to (16b),  $k_{13}(x, y, t)$  can be written as

$$k_{13}(x, y, t) = \frac{\exp(\gamma x)}{\kappa - 1} \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{13}(\rho, x) \exp(i\rho(y-t)) d\rho, \quad (86a)$$

$$\phi_{13}(\rho, x) = \sum_{j=1}^2 (i\rho N_j(\kappa + 1) + s_j(3 - \kappa)) \psi_j(\rho) \exp(s_j x), \quad (86b)$$

where  $s_j$  ( $j = 1, 2, 3, 4$ ),  $N_j$  and  $\psi_j(\rho)$  are given by (13), (15) and (19), respectively. Changing the limits of integration in (86a) and taking the limit as  $y \rightarrow 0$ , and rearranging,  $k_{13}(x, 0, t)$  can be written as

$$k_{13}(x, 0, t) = \frac{\exp(\gamma x)}{\kappa - 1} \frac{1}{2\pi} \int_0^{\infty} \left\{ K_{131}(\rho, x) \cos(\rho t) - K_{132}(\rho, x) \sin(\rho t) \right\} d\rho, \quad (87)$$

where,

$$K_{131}(\rho, x) = \phi_{13}(\rho, x) + \phi_{13}(-\rho, x), \quad (88a)$$

$$K_{132}(\rho, x) = i(\phi_{13}(\rho, x) - \phi_{13}(-\rho, x)). \quad (88b)$$

In order to extract the singular terms we expand  $K_{131}(\rho, x)$  and  $K_{132}(\rho, x)$  into series as  $\rho \rightarrow \infty$ . Following asymptotic expansions are obtained by using MAPLE,

$$K_{131}^\infty(\rho, x) = \left\{ d_{11}^* \rho + d_{10} + \frac{d_{11}}{\rho} + \frac{d_{12}}{\rho^2} + \cdots + \frac{d_{112}}{\rho^{12}} \right\} \exp(-(\gamma/2 + \rho)x), \quad (89a)$$

$$K_{132}^\infty(\rho, x) = \left\{ d_{21}^* \rho + d_{20} + \frac{d_{21}}{\rho} + \frac{d_{22}}{\rho^2} + \cdots + \frac{d_{212}}{\rho^{12}} \right\} \exp(-(\gamma/2 + \rho)x), \quad (89a)$$

where the coefficients are functions of  $x$ . The leading terms are given by,

$$d_{11}^* = -\frac{4(\kappa-1)\sin(\delta_3 x/2)}{\delta_3}, \quad d_{21}^* = -\frac{4\eta(\kappa-1)\sin(\delta_3 x/2)}{\delta_3}, \quad (90a,b)$$

and  $\delta_3$  by (12b). Subtracting the asymptotic expansions from the integrands in (87), using integration cutoff points for the infinite integrals and evaluating some of the integrals in closed form, after some manipulations (87) is reduced to,

$$k_{13}(x, 0, t) = \exp(\gamma x) \left\{ h_{13s}(x, t) + h_{13f}(x, t) \right\}, \quad (91)$$

$$h_{13s}(x, t) = \frac{1}{2(\kappa-1)} \frac{1}{\pi} \left\{ r_{131}(x, t) + r_{132}(x, t) \right\}, \quad (92a)$$

$$\begin{aligned} h_{13f}(x, t) = & \frac{1}{2(\kappa-1)} \frac{1}{\pi} \left\{ J_{131}(x, t) + J_{132}(x, t) + \right. \\ & + \frac{d_{11} \exp(-\gamma x/2)}{2} \left[ \Gamma(0, (x-it)A_{131}) + \Gamma(0, (x+it)A_{131}) \right] \\ & \left. + d_{21} \exp(-\gamma x/2) \arctan(-t/x) \right\}, \end{aligned} \quad (92b)$$

where  $A_{131}$  is an integration cutoff point and  $\Gamma(,)$  is the incomplete gamma function, [10] and

$$r_{131}(x, t) = \left\{ \frac{x^2 - t^2}{(x^2 + t^2)^2} d_{11}^* + \frac{x}{x^2 + t^2} d_{10} \right\} \exp(-\gamma x/2), \quad (93a)$$

$$r_{132}(x, t) = \left\{ \frac{-2tx}{(x^2 + t^2)^2} d_{21}^* - \frac{t}{x^2 + t^2} d_{20} \right\} \exp(-\gamma x/2), \quad (93b)$$

$$\begin{aligned} J_{131}(x, t) = & \int_0^{A_{131}} K_{131}^*(\rho, x) \cos(\rho t) d\rho \\ & + \int_{A_{131}}^{\infty} (K_{131}(\rho, x) - K_{131}^{\infty}(\rho, x)) \cos(\rho t) d\rho \\ & + \int_{A_{131}}^{\infty} \Gamma_{131}^{\infty}(\rho, x) \cos(\rho t) d\rho, \end{aligned} \quad (93c)$$

$$\begin{aligned} J_{132}(\rho, x) = & - \int_0^{A_{132}} K_{132}^*(\rho, x) \sin(\rho t) d\rho \\ & - \int_{A_{132}}^{\infty} (K_{132}(\rho, x) - K_{132}^{\infty}(\rho, x)) \sin(\rho t) d\rho \\ & - \int_{A_{132}}^{\infty} \Gamma_{132}^{\infty}(\rho, x) \sin(\rho t) d\rho, \end{aligned} \quad (93d)$$

where  $A_{132}$  is another integration cutoff point and the remaining terms are given by,

$$K_{131}^*(\rho, x) = K_{131}(\rho, x) - (d_{11}^* \rho + d_{10}) \exp(-(\gamma/2 + \rho)x), \quad (94a)$$

$$\Gamma_{131}^{\infty}(\rho, x) = \left\{ \frac{d_{12}}{\rho^2} + \frac{d_{13}}{\rho^3} + \cdots + \frac{d_{112}}{\rho^{12}} \right\} \exp(-(\gamma/2 + \rho)x), \quad (94b)$$

$$K_{132}^*(\rho, x) = K_{132}(\rho, x) - (d_{21}^* \rho + d_{20} + d_{21}/\rho) \exp(-(\gamma/2 + \rho)x), \quad (94c)$$

$$\Gamma_{132}^{\infty}(\rho, x) = \left\{ \frac{d_{22}}{\rho^2} + \frac{d_{23}}{\rho^3} + \cdots + \frac{d_{212}}{\rho^{12}} \right\} \exp(-(\gamma/2 + \rho)x). \quad (94d)$$

Second integrals in (93c) and (93d) will be neglected in numerical computation for sufficiently large values of  $A_{131}$  and  $A_{132}$ . Third integrals in (93c) and (93d) are evaluated in closed form. The expressions used in the evaluation of these integrals are given in Appendix A. Also note that,

$$\lim_{(x,t) \rightarrow 0} h_{13s}(x,t) = \frac{1}{\pi} \left\{ \frac{2xt^2}{(x^2 + t^2)^2} - \eta \frac{2t^3}{(x^2 + t^2)^2} \right\}. \quad (95)$$

This term becomes singular as  $x$  and  $t$  simultaneously approach zero, and is the standard expression obtained for the homogeneous materials (see, for example, equation (23b) in Dag and Erdogan [8]).

### 3.3 $k_{22}(x, y, t)$

We first express  $k_{22}(x, y, t)$  as follows

$$k_{22}(x, y, t) = k_{22}^{(i)}(x, y, t) + k_{22}^{(h)}(x, y, t), \quad (96)$$

where  $k_{22}^{(i)}$  is obtained from the infinite plane solution and  $k_{22}^{(h)}$  is obtained from the half-plane ( $x > 0$ ) solution. Referring to (45e)  $k_{22}^{(i)}(x, y, t)$  can be written as

$$k_{22}^{(i)}(x, y, t) = (\kappa + 1) \frac{\exp(\gamma x)}{4\pi} \int_{-\infty}^{\infty} \phi_{22}^{(i)}(\omega, y) \exp(i\omega(x - t)) d\omega, \quad (97a)$$

$$\phi_{22}^{(i)}(\omega, y) = \sum_{j=3}^4 (n_j + i\omega A_j) Z_j(\omega) \exp(n_j y), \quad (97b)$$

where  $n_j$ , ( $j = 1, 2, 3, 4$ ),  $A_j$  and  $Z_j$  are given by (23), (26) and (48), respectively.

Changing the limits of integration (97a) can be written as,

$$k_{22}^{(i)}(x, y, t) = (\kappa + 1) \frac{\exp(\gamma x)}{4\pi} \int_0^{\infty} \left\{ K_{221}^{(i)}(\omega, y) \cos(\omega(x - t)) + K_{222}^{(i)}(\omega, y) \sin(\omega(x - t)) \right\} d\omega, \quad (98)$$

where,

$$K_{221}^{(i)}(\omega, y) = \phi_{22}^{(i)}(\omega, y) + \phi_{22}^{(i)}(-\omega, y), \quad (99a)$$

$$K_{222}^{(i)}(\omega, y) = i \left( \phi_{22}^{(i)}(\omega, y) - \phi_{22}^{(i)}(-\omega, y) \right). \quad (99b)$$

In order to extract the singular terms we expand  $K_{221}^{(i)}$  and  $K_{222}^{(i)}$  into series as  $\omega \rightarrow \infty$ .

Following asymptotic expansions are obtained by using MAPLE

$$K_{221}^{(i)\infty}(\omega, y) = \left\{ \frac{k_{11}}{\omega} + \frac{k_{13}}{\omega^3} + \cdots + \frac{k_{111}}{\omega^{11}} + O\left(\frac{1}{\omega^{13}}\right) \right\} \exp(-\omega y), \quad (100a)$$

$$K_{222}^{(i)\infty}(\omega, y) = \left\{ k_{20} + \frac{k_{22}}{\omega^2} + \cdots + \frac{k_{212}}{\omega^{12}} + O\left(\frac{1}{\omega^{14}}\right) \right\} \exp(-\omega y), \quad (100b)$$

where the leading terms are

$$k_{20} = -\frac{4}{\kappa+1}, \quad k_{11} = \frac{2\gamma}{\kappa+1}. \quad (101a,b)$$

Subtracting the asymptotic expansions from the integrands in (98), using integration cutoff points for the infinite integrals, evaluating some integrals in closed form and taking the limit as  $y \rightarrow 0$ , and after some manipulations  $k_{22}^{(i)}(x, y, t)$  is reduced to,

$$k_{22}^{(i)}(x, 0, t) = \exp(\gamma x) \left\{ \frac{1}{\pi} \frac{1}{t-x} + (\kappa+1) \frac{1}{4\pi} \left[ J_{221}^{(i)}(x, t) + J_{222}^{(i)}(x, t) - k_{11} \ln(A_{221}^{(i)}|t-x|) \right] \right\}, \quad (102)$$

where,

$$\begin{aligned} J_{221}^{(i)}(x, t) &= \int_0^{A_{221}^{(i)}} K_{221}^{(i)}(\omega, 0) \cos(\omega(x-t)) d\omega \\ &+ \int_{A_{221}^{(i)}}^{\infty} \left[ K_{221}^{(i)}(\omega, 0) - K_{221}^{(i)\infty}(\omega, 0) \right] \cos(\omega(x-t)) d\omega \\ &+ \int_{A_{221}^{(i)}}^{\infty} \left[ K_{221}^{(i)\infty}(\omega, 0) - k_{11}/\omega \right] \cos(\omega(x-t)) d\omega \\ &- k_{11} \left\{ \gamma_0 + \int_0^{A_{221}^{(i)}} \frac{\cos(\alpha) - 1}{\alpha} d\alpha \right\}, \end{aligned} \quad (103a)$$

$$\begin{aligned}
J_{222}^{(i)}(x, t) &= \int_0^{A_{222}^{(i)}} \left[ K_{222}^{(i)}(\omega, 0) - k_{20} \right] \sin(\omega(x - t)) d\omega \\
&+ \int_{A_{222}^{(i)}}^{\infty} \left[ K_{222}^{(i)}(\omega, 0) - K_{222}^{(i)\infty}(\omega, 0) \right] \sin(\omega(x - t)) d\omega \\
&+ \int_{A_{222}^{(i)}}^{\infty} \left[ K_{222}^{(i)\infty}(\omega, 0) - k_{20} \right] \sin(\omega(x - t)) d\omega,
\end{aligned} \tag{103b}$$

$A_{221}^{(i)}$  and  $A_{222}^{(i)}$  are integration cutoff points and  $\gamma_0$  is the Euler number, [10]. Second integrals in (103a) and (103b) will be neglected in numerical computation for sufficiently large values of integration cutoff points. Third integrals in (103a) and (103b) are evaluated in closed form. The expressions used in the evaluation of these integrals are given in Appendix A. Referring to (54e) and (61a),  $k_{22}^{(h)}(x, y, t)$  can be written as follows

$$k_{22}^{(h)}(x, y, t) = (\kappa + 1) \frac{\exp(\gamma x)}{2} \int_0^{\infty} K_{22}^{(h)}(\alpha, t, x) \cos(\alpha y) d\alpha, \tag{104}$$

where

$$K_{22}^{(h)}(\alpha, t, x) = \sum_{j=3}^4 (\alpha + H_j t_j) G_j^*(\alpha, t) \exp(t_j x + (\gamma/2 - \lambda_1)t), \tag{105a}$$

$t_j$ , ( $j = 1, 2, 3, 4$ ),  $H_j(\alpha)$  and  $G_j^*(\alpha, t)$  are given by (53), (55) and (60), respectively. In order to extract the singular terms we expand  $K_{22}^{(h)}(\alpha, t, x)$  into series as  $\alpha$  tends to infinity as follows:

$$\begin{aligned}
K_{22}^{(h)\infty}(\alpha, t, x) &= \left\{ m_2^* \alpha^2 + m_1^* \alpha + m_0 + \frac{m_1}{\alpha} + \frac{m_2}{\alpha^2} + \cdots + \frac{m_7}{\alpha^7} + O\left(\frac{1}{\alpha^8}\right) \right\} \times \\
&\times \exp(\gamma(t - x)/2 - (t + x)\alpha),
\end{aligned} \tag{106}$$

where the coefficients of the expansion are also functions of  $x$  and  $t$ . The leading term is given by,

$$m_2^* = \frac{16}{\pi} \frac{1}{\kappa+1} \frac{\sin(\delta_1 x/2)\sin(\delta_1 t/2)}{\delta_1^2} \quad (107)$$

and  $\delta_1$  is given by (22b). Subtracting  $K_{22}^{(h)\infty}$  from the integrand in (104), evaluating some of the integrals in closed form, using an integration cutoff point  $A_{22}^{(h)}$  for the infinite integral and taking the limit as  $y \rightarrow 0$ , after some manipulations  $k_{22}^{(h)}$  can be written as,

$$k_{22}^{(h)}(x, 0, t) = (\kappa+1) \frac{\exp(\gamma x)}{2} \left\{ r_{22}^{(h)}(x, t) + J_{22}^{(h)}(x, t) - m_1 \exp(\gamma(t-x)/2) \text{Ei}\left(-A_{22}^{(h)}(t+x)\right) \right\}, \quad (108)$$

where  $\text{Ei}()$  is the exponential integral [10] and,

$$r_{22}^{(h)}(x, t) = \left\{ \frac{2m_2^*}{(t+x)^3} + \frac{m_1^*}{(t+x)^2} + \frac{m_0}{t+x} \right\} \exp(\gamma(t-x)/2), \quad (109a)$$

$$\begin{aligned} J_{22}^{(h)}(x, t) &= \int_0^{A_{22}^{(h)}} K_{22}^{(h)*}(\alpha, t, x) d\alpha \\ &+ \int_{A_{22}^{(h)}}^{\infty} \left[ K_{22}^{(h)}(\alpha, t, x) - K_{22}^{(h)\infty}(\alpha, t, x) \right] d\alpha \\ &+ \int_{A_{22}^{(h)}}^{\infty} \Gamma_{22}^{(h)\infty}(\alpha, t, x) d\alpha, \end{aligned} \quad (109b)$$

$$K_{22}^{(h)*}(\alpha, t, x) = K_{22}^{(h)}(\alpha, t, x) - (m_2^* \alpha^2 + m_1^* \alpha + m_0) \exp(\gamma(t-x)/2 - (t+x)\alpha), \quad (110a)$$

$$\Gamma_{22}^{(h)\infty}(\alpha, t, x) = \left\{ \frac{m_2}{\alpha^2} + \frac{m_3}{\alpha^3} + \cdots + \frac{m_7}{\alpha^7} \right\} \exp(\gamma(t-x)/2 - (t+x)\alpha). \quad (110b)$$

Second integral in (109b) will be neglected in numerical computation for a sufficiently large value of  $A_{22}^{(h)}$ . Third integral in (109b) is evaluated in closed form. The expression used in the evaluation of this integral is given in Appendix A.  $k_{22}^{(i)}(x, 0, t)$  and  $k_{22}^{(h)}(x, 0, t)$  are given by equations (102) and (108) respectively. Adding these two equations  $k_{22}(x, 0, t)$  is expressed in the following form,

$$k_{22}(x, 0, t) = \exp(\gamma x) \left\{ \frac{1}{\pi} \frac{1}{t-x} + h_{22s}(x, t) + h_{22f}(x, t) \right\}, \quad (111)$$

where,

$$h_{22s}(x, t) = \frac{\kappa + 1}{2} r_{22}^{(h)}(x, t), \quad (112a)$$

$$h_{22f}(x, t) = (\kappa + 1) \frac{1}{4\pi} \left\{ -k_{11} \ln \left( A_{221}^{(i)} |t-x| \right) + J_{221}^{(i)}(x, t) + J_{222}^{(i)}(x, t) \right\} + \\ + \frac{\kappa + 1}{2} \left\{ J_{22}^{(h)}(x, t) - m_1 \exp(\gamma(t-x)/2) \operatorname{Ei} \left( -A_{22}^{(h)}(t+x) \right) \right\}. \quad (112b)$$

The terms in (112) are given explicitly by (103) and (109). Note that the first term in (111) is the Cauchy singularity associated with a crack in infinite plane, as for the second term, we can write,

$$\lim_{(x,t) \rightarrow 0} h_{22s}(x, t) = \frac{1}{\pi} \left( \frac{1}{t+x} + \frac{2t}{(t+x)^2} - \frac{4t^2}{(t+x)^3} \right). \quad (113)$$

This term becomes singular as  $x$  and  $t$  simultaneously approach zero, and is the standard expression found for edge cracks in homogeneous materials. (see for example, equation (23a) in Dag and Erdogan [8]). Also, note that if the medium is homogeneous (i.e.,  $\gamma = 0$ ),  $h_{22f} = 0$  in (111) and  $h_{11f} = 0$  in (83) and  $k_{11}(x, 0, t) = k_{22}(x, 0, t)$ , but if  $\gamma \neq 0$ , this equality is not valid and consequently for graded materials  $k_{11}(x, 0, t) \neq k_{22}(x, 0, t)$ .

### 3.4 $k_{23}(x, y, t)$

Referring to (16c),  $k_{23}(x, y, t)$  can be written as,

$$k_{23}(x, y, t) = \frac{\exp(\gamma x)}{2\pi} \int_{-\infty}^{\infty} \phi_{23}(\rho, x) \exp(i\rho(y-t)) d\rho, \quad (114a)$$

$$\phi_{23}(\rho, x) = \sum_{j=1}^2 (i\rho + N_j s_j) \psi_j(\rho) \exp(s_j x), \quad (114b)$$

where  $s_j$ , ( $j = 1, 2, 3, 4$ ),  $N_j$  and  $\psi_j(\rho)$  are given by (13), (15) and (19), respectively. Changing the limits of integration in (114a) and taking the limit as  $y \rightarrow 0$ , and rearranging,  $k_{23}(x, 0, t)$  can be written as

$$k_{23}(x, 0, t) = \frac{\exp(\gamma x)}{2\pi} \int_0^\infty \left\{ K_{231}(\rho, x) \cos(\rho t) - K_{232}(\rho, x) \sin(\rho t) \right\} d\rho, \quad (115)$$

where,

$$K_{231}(\rho, x) = \phi_{23}(\rho, x) + \phi_{23}(-\rho, x), \quad (116a)$$

$$K_{232}(\rho, x) = i(\phi_{23}(\rho, x) - \phi_{23}(-\rho, x)). \quad (116b)$$

In order to extract the singular terms we expand  $K_{231}(\rho, x)$  and  $K_{232}(\rho, x)$  into series as  $\rho \rightarrow \infty$ . Following asymptotic expansions are obtained by using MAPLE,

$$K_{231}^\infty(\rho, x) = \left\{ e_{11}^* \rho + e_{10} + \frac{e_{11}}{\rho} + \frac{e_{12}}{\rho^2} + \cdots + \frac{e_{112}}{\rho^{12}} \right\} \exp(-(\gamma/2 + \rho)x), \quad (117a)$$

$$K_{232}^\infty(\rho, x) = \left\{ e_{21}^* \rho + e_{20} + \frac{e_{21}}{\rho} + \frac{e_{22}}{\rho^2} + \cdots + \frac{e_{212}}{\rho^{12}} \right\} \exp(-(\gamma/2 + \rho)x), \quad (117b)$$

where the coefficients are functions of  $x$ . The leading terms are given by,

$$e_{11}^* = -\frac{4\eta \sin(\delta_3 x/2)}{\delta_3}, \quad e_{21}^* = -\frac{4\sin(\delta_3 x/2)}{\delta_3}, \quad (118a,b)$$

and  $\delta_3$  by (12b). Subtracting the asymptotic expansions from the integrands in (115), using integration cutoff points for the infinite integrals and evaluating some of the integrals in closed form, after some manipulations (115) is reduced to,

$$k_{23}(x, 0, t) = \exp(\gamma x) \left\{ h_{23s}(x, t) + h_{23f}(x, t) \right\}, \quad (119)$$

$$h_{23s}(x, t) = \frac{1}{2\pi} \left\{ r_{231}(x, t) + r_{232}(x, t) \right\}, \quad (120a)$$

$$\begin{aligned} h_{23f}(x, t) = & \frac{1}{2\pi} \left\{ J_{231}(x, t) + J_{232}(x, t) + \right. \\ & + \frac{e_{11}\exp(-\gamma x/2)}{2} [\Gamma(0, (x - it)A_{231}) + \Gamma(0, (x + it)A_{231})] \\ & \left. + e_{21}\exp(-\gamma x/2)\arctan(-t/x) \right\}, \end{aligned} \quad (120b)$$

where  $A_{231}$  is an integration cutoff point,  $\Gamma(,)$  is the incomplete gamma function, [10],

$$r_{231}(x, t) = \left\{ \frac{x^2 - t^2}{(x^2 + t^2)^2} e_{11}^* + \frac{x}{x^2 + t^2} e_{10} \right\} \exp(-\gamma x/2), \quad (121a)$$

$$r_{232}(x, t) = \left\{ \frac{-2tx}{(x^2 + t^2)^2} e_{21}^* - \frac{t}{x^2 + t^2} e_{20} \right\} \exp(-\gamma x/2), \quad (121b)$$

$$\begin{aligned} J_{231}(x, t) = & \int_0^{A_{231}} K_{231}^*(\rho, x) \cos(\rho t) d\rho \\ & + \int_{A_{231}}^{\infty} (K_{231}(\rho, x) - K_{231}^{\infty}(\rho, x)) \cos(\rho t) d\rho \\ & + \int_{A_{231}}^{\infty} \Gamma_{231}^{\infty}(\rho, x) \cos(\rho t) d\rho, \end{aligned} \quad (121c)$$

$$\begin{aligned} J_{232}(\rho, x) = & - \int_0^{A_{232}} K_{232}^*(\rho, x) \sin(\rho t) d\rho \\ & - \int_{A_{232}}^{\infty} (K_{232}(\rho, x) - K_{232}^{\infty}(\rho, x)) \sin(\rho t) d\rho \\ & - \int_{A_{232}}^{\infty} \Gamma_{232}^{\infty}(\rho, x) \sin(\rho t) d\rho, \end{aligned} \quad (121d)$$

and  $A_{232}$  is another integration cutoff point. The remaining terms are given by

$$K_{231}^*(\rho, x) = K_{231}(\rho, x) - (e_{11}^* \rho + e_{10}) \exp(-(\gamma/2 + \rho)x), \quad (122a)$$

$$\Gamma_{231}^{\infty}(\rho, x) = \left\{ \frac{e_{12}}{\rho^2} + \frac{e_{13}}{\rho^3} + \cdots + \frac{e_{112}}{\rho^{12}} \right\} \exp(-(\gamma/2 + \rho)x), \quad (122b)$$

$$K_{232}^*(\rho, x) = K_{232}(\rho, x) - (e_{21}^* \rho + e_{20} + e_{21}/\rho) \exp(-(\gamma/2 + \rho)x), \quad (122c)$$

$$\Gamma_{232}^{\infty}(\rho, x) = \left\{ \frac{e_{22}}{\rho^2} + \frac{e_{23}}{\rho^3} + \cdots + \frac{e_{212}}{\rho^{12}} \right\} \exp(-(\gamma/2 + \rho)x). \quad (123d)$$

Second integrals in (121c) and (121d) will be neglected in numerical computation for sufficiently large values of  $A_{231}$  and  $A_{232}$ . Third integrals in (121c) and (121d) are evaluated in closed form. The expressions used in the evaluation of these integrals are given in Appendix A. Also note that

$$\lim_{(x,t) \rightarrow 0} h_{23s}(x, t) = \frac{1}{\pi} \left\{ \eta \frac{2xt^2}{(x^2 + t^2)^2} - \frac{2tx^2}{(x^2 + t^2)^2} \right\}. \quad (124)$$

This term becomes singular as  $x$  and  $t$  simultaneously approach zero, and is the standard expression obtained for the homogeneous materials (see, for example, equation (23c) in Dag and Erdogan [8]).

### 3.5 $k_{31}(x, y, t)$

We first express  $k_{31}(x, y, t)$  as follows:

$$k_{31}(x, y, t) = k_{31}^{(i)}(x, y, t) + k_{31}^{(h)}(x, y, t), \quad (125)$$

where  $k_{31}^{(i)}$  and  $k_{31}^{(h)}$  are obtained from the infinite plane and half-plane ( $x > 0$ ) solutions respectively. Referring to (25f),  $k_{31}^{(i)}(x, y, t)$  can be expressed as,

$$k_{31}^{(i)}(x, y, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \phi_{31}^{(i)}(\omega, y) \exp(i\omega(x - t)) d\omega, \quad (126a)$$

$$\phi_{31}^{(i)}(\omega, y) = \sum_{j=3}^4 P_j(\omega) n_j \exp(n_j y), \quad (126b)$$

$n_j$ , ( $j = 1, 2, 3, 4$ ),  $P_j$  are given by (23) and (29), respectively. Changing the limits of integration, (126a) can be written as

$$k_{31}^{(i)}(x, y, t) = \frac{1}{\pi} \int_0^\infty \left\{ K_{311}^{(i)}(\omega, y) \cos(\omega(x - t)) + K_{312}^{(i)}(\omega, y) \sin(\omega(x - t)) \right\} d\omega \quad (127)$$

where

$$K_{311}^{(i)}(\omega, y) = \phi_{31}^{(i)}(\omega, y) + \phi_{31}^{(i)}(-\omega, y), \quad (128a)$$

$$K_{312}^{(i)}(\omega, y) = i \left( \phi_{31}^{(i)}(\omega, y) - \phi_{31}^{(i)}(-\omega, y) \right). \quad (128b)$$

In order to extract the singular terms we expand  $K_{311}^{(i)}$  and  $K_{312}^{(i)}$  into series as  $\omega \rightarrow \infty$ . Following asymptotic expansions are obtained by using MAPLE

$$K_{311}^{(i)\infty}(\omega, y) = \left\{ g_{11}^* \omega + g_{10} + \frac{g_{11}}{\omega} + \frac{g_{12}}{\omega^2} + \cdots + \frac{g_{112}}{\omega^{12}} \right\} \exp(-\omega y), \quad (129a)$$

$$K_{312}^{(i)\infty}(\omega, y) = \left\{ g_{21}^* \omega + g_{20} + \frac{g_{21}}{\omega} + \frac{g_{22}}{\omega^2} + \cdots + \frac{g_{212}}{\omega^{12}} \right\} \exp(-\omega y), \quad (129b)$$

where the leading terms are given in Appendix B. Note that the coefficients of the expansion are functions of  $y$ . Subtracting the asymptotic expansions from the integrands in (127), using integration cutoff points for the infinite integrals, evaluating some of the integrals in closed form and taking the limit as  $x \rightarrow 0$ , after some manipulations  $k_{31}^{(i)}(x, y, t)$  is reduced to

$$\begin{aligned} k_{31}^{(i)}(0, y, t) = & \frac{1}{\pi} \left\{ r_{31}^{(i)}(y, t) + J_{311}^{(i)}(y, t) + J_{312}^{(i)}(y, t) \right. \\ & + \frac{g_{11}}{2} \left[ \Gamma(0, (y - it) A_{311}^{(i)}) + \Gamma(0, (y + it) A_{311}^{(i)}) \right] \\ & \left. + g_{21} \arctan(-t/y) \right\}, \end{aligned} \quad (130)$$

where  $A_{311}^{(i)}$  is an integration cutoff point,  $\Gamma(,)$  is the incomplete gamma function [10],

$$r_{31}^{(i)}(y, t) = \frac{y^2 - t^2}{(y^2 + t^2)^2} g_{11}^* + \frac{y}{y^2 + t^2} g_{10} - \frac{2ty}{(y^2 + t^2)^2} g_{21}^* - \frac{t}{y^2 + t^2} g_{20}, \quad (131a)$$

$$\begin{aligned} J_{311}^{(i)}(y, t) = & \int_0^{A_{311}^{(i)}} K_{311}^{(i)*}(\omega, y) \cos(\omega t) d\omega \\ & + \int_{A_{311}^{(i)}}^{\infty} \left\{ K_{311}^{(i)}(\omega, y) - K_{311}^{(i)\infty}(\omega, y) \right\} \cos(\omega t) d\omega \\ & + \int_{A_{311}^{(i)}}^{\infty} \Gamma_{311}^{(i)\infty}(\omega, y) \cos(\omega t) d\omega, \end{aligned} \quad (131b)$$

$$\begin{aligned} J_{312}^{(i)}(y, t) = & - \int_0^{A_{312}^{(i)}} K_{312}^{(i)*}(\omega, y) \sin(\omega t) d\omega \\ & - \int_{A_{312}^{(i)}}^{\infty} \left\{ K_{312}^{(i)}(\omega, y) - K_{312}^{(i)\infty}(\omega, y) \right\} \sin(\omega t) d\omega \\ & - \int_{A_{312}^{(i)}}^{\infty} \Gamma_{312}^{(i)\infty}(\omega, y) \sin(\omega t) d\omega, \end{aligned} \quad (131c)$$

$A_{312}^{(i)}$  is another integration cutoff point and the remaining terms are given by

$$K_{311}^{(i)*}(\omega, y) = K_{311}^{(i)}(\omega, y) - (g_{11}^* \omega + g_{10}) \exp(-\omega y), \quad (132a)$$

$$\Gamma_{311}^{(i)\infty}(\omega, y) = \left\{ \frac{g_{12}}{\omega^2} + \frac{g_{13}}{\omega^3} + \cdots + \frac{g_{112}}{\omega^{12}} \right\} \exp(-\omega y), \quad (132b)$$

$$K_{312}^{(i)*}(\omega, y) = K_{312}^{(i)}(\omega, y) - (g_{21}^* \omega + g_{20} + g_{21}/\omega) \exp(-\omega y), \quad (132c)$$

$$\Gamma_{312}^{(i)\infty}(\omega, y) = \left\{ \frac{g_{22}}{\omega^2} + \frac{g_{23}}{\omega^3} + \cdots + \frac{g_{212}}{\omega^{12}} \right\} \exp(-\omega y), \quad (132d)$$

Second integrals in (131b) and (131c) will be neglected in numerical computation for sufficiently large values of  $A_{311}^{(i)}$  and  $A_{312}^{(i)}$ . Third integrals in (131b) and (131c) are

evaluated in closed form. The expressions used in the evaluation of these integrals are given in Appendix A. Referring to (35f) and (41a),  $k_{31}^{(h)}(x, y, t)$  can be written as follows:

$$k_{31}^{(h)}(x, y, t) = 2 \int_0^\infty K_{31}^{(h)}(\alpha, t, x) \sin(\alpha y) d\alpha, \quad (133)$$

where

$$K_{31}^{(h)}(\alpha, t, x) = -\alpha \sum_{j=3}^4 B_j^*(\alpha, t) \exp(p_j x + (\gamma/2 - \lambda_1)t), \quad (134)$$

$p_j$ , ( $j = 1, 2, 3, 4$ ) and  $B_j^*(\alpha, t)$  are given by (34) and (41), respectively. In order to extract the singular terms we expand  $K_{31}^{(h)}(\alpha, t, 0)$  into a series as  $\alpha$  tends to infinity as follows:

$$K_{31}^{(h)\infty}(\alpha, t, 0) = \left\{ i_1^* \alpha + i_0 + \frac{i_1}{\alpha} + \frac{i_2}{\alpha^2} + \cdots + \frac{i_7}{\alpha^7} \right\} \exp((\gamma/2 - \alpha)t) \quad (135)$$

where the coefficients of the expansion are functions of  $t$ . The leading term is given by

$$i_1^* = -\frac{2}{\pi} \frac{\kappa \sin(\delta_1 t/2)}{(\kappa + 1)\delta_1}, \quad (136)$$

$\delta_1$  by (22b). Subtracting  $K_{31}^{(h)\infty}$  from the integrand in (133), evaluating some of the integrals in closed form, using an integration cutoff point for the infinite integral and taking the limit as  $x \rightarrow 0$ , after some manipulations  $k_{31}^{(h)}$  can be written as

$$k_{31}^{(h)}(0, y, t) = 2 \left\{ r_{31}^{(h)}(y, t) + i_1 \exp(\gamma t/2) \arctan(y/t) + J_{31}^{(h)}(y, t) \right\}, \quad (137)$$

$$r_{31}^{(h)}(y, t) = \left\{ \frac{2yt}{(y^2 + t^2)^2} i_1^* + \frac{y}{y^2 + t^2} i_0 \right\} \exp(\gamma t/2), \quad (138a)$$

$$\begin{aligned}
J_{31}^{(h)}(y, t) &= \int_0^{A_{31}^{(h)}} K_{31}^{(h)*}(\alpha, t) \sin(\alpha y) d\alpha \\
&+ \int_{A_{31}^{(h)}}^{\infty} \left[ K_{31}^{(h)}(\alpha, t, 0) - K_{31}^{(h)\infty}(\alpha, t, 0) \right] \sin(\alpha y) d\alpha \\
&+ \int_{A_{31}^{(h)}}^{\infty} \Gamma_{31}^{(h)\infty}(\alpha, t) \sin(\alpha y) d\alpha.
\end{aligned} \tag{138b}$$

In (138)  $A_{31}^{(h)}$  is an integration cutoff point and the remaining terms are given by

$$K_{31}^{(h)*}(\alpha, t) = K_{31}^{(h)}(\alpha, t, 0) - (i_1^* \alpha + i_0 + i_1/\alpha) \exp((\gamma/2 - \alpha)t), \tag{139a}$$

$$\Gamma_{31}^{(h)\infty}(\alpha, t) = \left\{ \frac{i_2}{\alpha^2} + \frac{i_3}{\alpha^3} + \cdots + \frac{i_7}{\alpha^7} \right\} \exp((\gamma/2 - \alpha)t). \tag{139b}$$

Note that second integral in (138b) will be neglected in numerical computation for a sufficiently large value of  $A_{31}^{(h)}$  and third integral is evaluated in closed form. The expression used in the evaluation of this integral is given in Appendix A.  $k_{31}^{(i)}(0, y, t)$  and  $k_{31}^{(h)}(0, y, t)$  are given by equations (130) and (137), respectively. Adding these two equations,  $k_{31}(0, y, t)$  is expressed in the following form,

$$k_{31}(0, y, t) = h_{31s}(y, t) + h_{31f}(y, t), \tag{140}$$

where,

$$h_{31s}(y, t) = \frac{1}{\pi} r_{31}^{(i)}(y, t) + 2r_{31}^{(h)}(y, t), \tag{141a}$$

$$\begin{aligned}
h_{31f}(y, t) &= \frac{1}{\pi} \left\{ J_{311}^{(i)}(y, t) + J_{312}^{(i)}(y, t) + \right. \\
&+ \frac{g_{11}}{2} \left[ \Gamma(0, (y - it) A_{311}^{(i)}) + \Gamma(0, (y + it) A_{311}^{(i)}) \right] + g_{21} \arctan(-t/y) \left. \right\} \\
&+ 2 \left\{ J_{31}^{(h)}(y, t) + i_1 \exp(\gamma t/2) \arctan(y/t) \right\}.
\end{aligned} \tag{141b}$$

The terms in (141) are given explicitly by (131) and (138). Also, note that,

$$\lim_{(y,t) \rightarrow 0} h_{31s}(y, t) = -\frac{4}{\pi} \frac{t^2 y}{(y^2 + t^2)^2}. \quad (142)$$

This term becomes singular as  $y$  and  $t$  simultaneously tend to zero, and is the standard expression found for homogeneous materials. (see, for example, equation (23d) in Dag and Erdogan [8]).

### 3.6 $k_{32}(x, y, t)$

We first express  $k_{32}(x, y, t)$  as follows,

$$k_{32}(x, y, t) = k_{32}^{(i)}(x, y, t) + k_{32}^{(h)}(x, y, t), \quad (143)$$

where  $k_{32}^{(i)}$  and  $k_{32}^{(h)}$  are obtained from the infinite plane and half-plane ( $x > 0$ ) solutions respectively. Referring to (45f)  $k_{32}^{(i)}(x, y, t)$  can be expressed as

$$k_{32}^{(i)}(x, y, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \phi_{32}^{(i)}(\omega, y) \exp(i\omega(x - t)) d\omega, \quad (144a)$$

$$\phi_{32}^{(i)}(\omega, y) = \sum_{j=3}^4 Z_j(\omega) n_j \exp(n_j y), \quad (144b)$$

where  $n_j$ , ( $j = 1, 2, 3, 4$ ) is given by (23) and  $Z_j$  is given by (48). Changing the limits of integration (144a) can be written as

$$k_{32}^{(i)}(x, y, t) = \frac{1}{\pi} \int_0^{\infty} \left\{ K_{321}^{(i)}(\omega, y) \cos(\omega(x - t)) + K_{322}^{(i)}(\omega, y) \sin(\omega(x - t)) \right\} d\omega, \quad (145)$$

where

$$K_{321}^{(i)}(\omega, y) = \phi_{32}^{(i)}(\omega, y) + \phi_{32}^{(i)}(-\omega, y), \quad (146a)$$

$$K_{312}^{(i)}(\omega, y) = i \left( \phi_{32}^{(i)}(\omega, y) - \phi_{32}^{(i)}(-\omega, y) \right). \quad (146b)$$

In order to extract the singular terms we expand  $K_{321}^{(i)}$  and  $K_{322}^{(i)}$  into series as  $\omega \rightarrow \infty$ .

Following asymptotic expansions are obtained by using MAPLE:

$$K_{321}^{(i)\infty}(\omega, y) = \left\{ l_{11}^* \omega + l_{10} + \frac{l_{11}}{\omega} + \frac{l_{12}}{\omega^2} + \cdots + \frac{l_{112}}{\omega^{12}} \right\} \exp(-\omega y), \quad (147a)$$

$$K_{322}^{(i)\infty}(\omega, y) = \left\{ l_{21}^* \omega + l_{20} + \frac{l_{21}}{\omega} + \frac{l_{22}}{\omega^2} + \cdots + \frac{l_{212}}{\omega^{12}} \right\} \exp(-\omega y), \quad (147b)$$

where the leading terms are given in Appendix B. Note that the coefficients of the expansion are functions of  $y$ . Subtracting the asymptotic expansions from the integrands in (145), using integration cutoff points for the infinite integrals, evaluating some integrals in closed form and taking the limit as  $x \rightarrow 0$ , after some manipulations  $k_{32}^{(i)}(x, y, t)$  is reduced to:

$$\begin{aligned} k_{32}^{(i)}(0, y, t) = & \frac{1}{\pi} \left\{ r_{32}^{(i)}(y, t) + J_{321}^{(i)}(y, t) + J_{322}^{(i)}(y, t) \right. \\ & + \frac{l_{11}}{2} \left[ \Gamma \left( 0, (y - it) A_{321}^{(i)} \right) + \Gamma \left( 0, (y + it) A_{321}^{(i)} \right) \right] \\ & \left. + l_{21} \arctan(-t/y) \right\}, \end{aligned} \quad (148)$$

where  $A_{321}^{(i)}$  is an integration cutoff point,  $\Gamma(,)$  is the incomplete gamma function,

$$r_{32}^{(i)}(y, t) = \frac{y^2 - t^2}{(y^2 + t^2)^2} l_{11}^* + \frac{y}{y^2 + t^2} l_{10} - \frac{2ty}{(y^2 + t^2)^2} l_{21}^* - \frac{t}{y^2 + t^2} l_{20}, \quad (149a)$$

$$\begin{aligned}
J_{321}^{(i)}(y, t) &= \int_0^{A_{321}^{(i)}} K_{321}^{(i)*}(\omega, y) \cos(\omega t) d\omega \\
&+ \int_{A_{321}^{(i)}}^{\infty} \left\{ K_{321}^{(i)}(\omega, y) - K_{321}^{(i)\infty}(\omega, y) \right\} \cos(\omega t) d\omega \\
&+ \int_{A_{321}^{(i)}}^{\infty} \Gamma_{321}^{(i)\infty}(\omega, y) \cos(\omega t) d\omega,
\end{aligned} \tag{149b}$$

$$\begin{aligned}
J_{322}^{(i)}(y, t) &= - \int_0^{A_{322}^{(i)}} K_{322}^{(i)*}(\omega, y) \sin(\omega t) d\omega \\
&- \int_{A_{322}^{(i)}}^{\infty} \left\{ K_{322}^{(i)}(\omega, y) - K_{322}^{(i)\infty}(\omega, y) \right\} \sin(\omega t) d\omega \\
&- \int_{A_{322}^{(i)}}^{\infty} \Gamma_{322}^{(i)\infty}(\omega, y) \sin(\omega t) d\omega,
\end{aligned} \tag{149c}$$

$A_{322}^{(i)}$  is another integration cutoff point and the remaining terms are given by

$$K_{321}^{(i)*}(\omega, y) = K_{321}^{(i)}(\omega, y) - (l_{11}^* \omega + l_{10}) \exp(-\omega y), \tag{150a}$$

$$\Gamma_{321}^{(i)\infty}(\omega, y) = \left\{ \frac{l_{12}}{\omega^2} + \frac{l_{13}}{\omega^3} + \cdots + \frac{l_{112}}{\omega^{12}} \right\} \exp(-\omega y), \tag{150b}$$

$$K_{322}^{(i)*}(\omega, y) = K_{322}^{(i)}(\omega, y) - (l_{21}^* \omega + l_{20} + l_{21}/\omega) \exp(-\omega y), \tag{150c}$$

$$\Gamma_{322}^{(i)\infty}(\omega, y) = \left\{ \frac{l_{22}}{\omega^2} + \frac{l_{23}}{\omega^3} + \cdots + \frac{l_{212}}{\omega^{12}} \right\} \exp(-\omega y), \tag{150d}$$

Second integrals in (149b) and (149c) will be neglected in numerical computation for sufficiently large values of  $A_{321}^{(i)}$  and  $A_{322}^{(i)}$ . Third integrals in (149b) and (149c) are evaluated in closed form. The expressions used in the evaluation of these integrals are given in Appendix A. Referring to (54f) and (61a),  $k_{32}^{(h)}(x, y, t)$  can be written as follows:

$$k_{32}^{(h)}(x, y, t) = 2 \int_0^{\infty} K_{32}^{(h)}(\alpha, t, x) \cos(\alpha y) d\alpha, \tag{151}$$

where

$$K_{32}^{(h)}(\alpha, t, x) = \alpha \sum_{j=3}^4 G_j^*(\alpha, t) \exp(p_j x + (\gamma/2 - \lambda_1)t), \quad (152)$$

$t_j$ , ( $j = 1, 2, 3, 4$ ) is given by (53) and  $G_j^*(\alpha, t)$  is given by (60). In order to extract the singular terms we expand  $K_{32}^{(h)}(\alpha, t, 0)$  into a series as  $\alpha$  tends to infinity as follows

$$K_{32}^{(h)\infty}(\alpha, t, 0) = \left\{ n_1^* \alpha + n_0 + \frac{n_1}{\alpha} + \frac{n_2}{\alpha^2} + \cdots + \frac{n_7}{\alpha^7} \right\} \exp((\gamma/2 - \alpha)t), \quad (153)$$

where the coefficients of the expansion are functions of  $t$ . The leading term is given by

$$n_1^* = -\frac{2}{\pi} \frac{\kappa \sin(\delta_1 t/2)}{(\kappa + 1)\delta_1}, \quad (154)$$

and  $\delta_1$  by (22b). Subtracting  $K_{32}^{(h)\infty}$  from the integrand in (151), evaluating some integrals in closed form, using an integration cutoff point for the infinite integral and taking the limit as  $x \rightarrow 0$ , after some manipulations  $k_{32}^{(h)}$  can be written as

$$k_{32}^{(h)}(0, y, t) = 2 \left\{ r_{32}^{(h)}(y, t) + J_{32}^{(h)}(y, t) + \right. \\ \left. + \frac{n_1}{2} \left[ \Gamma(0, (t + iy)A_{32}^{(h)}) + \Gamma(0, (t - iy)A_{32}^{(h)}) \right] \exp(\gamma t/2) \right\}, \quad (155)$$

where,

$$r_{32}^{(h)}(y, t) = \left\{ \frac{t^2 - y^2}{(y^2 + t^2)^2} n_1^* + \frac{t}{y^2 + t^2} n \right\} \exp(\gamma t/2), \quad (156a)$$

$$J_{32}^{(h)}(y, t) = \int_0^{A_{32}^{(h)}} K_{32}^{(h)*}(\alpha, t) \cos(\alpha y) d\alpha \\ + \int_{A_{32}^{(h)}}^{\infty} \left[ K_{32}^{(h)}(\alpha, t, 0) - K_{32}^{(h)\infty}(\alpha, t, 0) \right] \cos(\alpha y) d\alpha \\ + \int_{A_{32}^{(h)}}^{\infty} \Gamma_{32}^{(h)\infty}(\alpha, t) \cos(\alpha y) d\alpha. \quad (156b)$$

$A_{32}^{(h)}$  is an integration cutoff point, and the remaining terms in (156b) are given by

$$K_{32}^{(h)*}(\alpha, t) = K_{32}^{(h)}(\alpha, t, 0) - (n_1^* \alpha + n_0) \exp((\gamma/2 - \alpha)t), \quad (157a)$$

$$\Gamma_{32}^{(h)\infty}(\alpha, t) = \left\{ \frac{n_2}{\alpha^2} + \frac{n_3}{\alpha^3} + \cdots + \frac{n_7}{\alpha^7} \right\} \exp((\gamma/2 - \alpha)t). \quad (157b)$$

Note that second integral in (156b) will be neglected in numerical computation for a sufficiently large value of  $A_{32}^{(h)}$  and third integral is evaluated in closed form. The expression used in the evaluation of this integral is given in Appendix A.  $k_{32}^{(i)}(0, y, t)$  and  $k_{32}^{(h)}(0, y, t)$  are given by equations (148) and (155), respectively. Adding these two equations,  $k_{32}(0, y, t)$  may be expressed in the following form,

$$k_{32}(0, y, t) = h_{32s}(y, t) + h_{32f}(y, t), \quad (158)$$

where

$$h_{32s}(y, t) = \frac{1}{\pi} r_{32}^{(i)}(y, t) + 2r_{32}^{(h)}(y, t), \quad (159a)$$

$$\begin{aligned} h_{32f}(y, t) = & \frac{1}{\pi} \left\{ J_{321}^{(i)}(y, t) + J_{322}^{(i)}(y, t) + \right. \\ & + \frac{l_{11}}{2} \left[ \Gamma(0, (y - it)A_{321}^{(i)}) + \Gamma(0, (y + it)A_{321}^{(i)}) \right] + l_{21} \arctan(-t/y) \Big\} \\ & + 2 \left\{ J_{32}^{(h)}(y, t) + \frac{n_1}{2} \left[ \Gamma(0, (t + iy)A_{32}^{(h)}) + \Gamma(0, (t - iy)A_{32}^{(h)}) \right] \exp(\gamma t/2) \right\}. \end{aligned} \quad (159b)$$

The terms in (159) are given explicitly by (149) and (156). Also, note that,

$$\lim_{(y,t) \rightarrow 0} h_{32s}(y, t) = \frac{4}{\pi} \frac{t^3}{(y^2 + t^2)^2}. \quad (160)$$

This term becomes singular as  $y$  and  $t$  simultaneously tend to zero, and it is the standard expression found for homogeneous materials. (see, for example, equation (23e) in Dag and Erdogan [8]).

### 3.7 $k_{33}(x, y, t)$

Referring to (16d),  $k_{33}(x, y, t)$  can be written as

$$k_{33}(x, y, t) = \frac{2}{\kappa + 1} \frac{1}{\pi} \int_{-\infty}^{\infty} \phi_{33}(\rho, x) \exp(i\rho(y - t)) d\rho, \quad (161a)$$

$$\phi_{33}(\rho, x) = i\rho \sum_{j=1}^2 \psi_j(\rho) \exp(s_j x), \quad (161b)$$

where,  $s_j$ , ( $j = 1, 2, 3, 4$ ) and  $\psi_j(\rho)$  are given by (13) and (19), respectively. Changing the limits of integration in (161a),  $k_{33}$  can be written as

$$k_{33}(x, y, t) = \frac{2}{\kappa + 1} \frac{1}{\pi} \int_0^{\infty} [K_{331}(\rho, x) \cos(\rho(y - t)) + K_{332}(\rho, x) \sin(\rho(y - t))] d\rho, \quad (162)$$

where,

$$K_{331}(\rho, x) = \phi_{33}(\rho, x) + \phi_{33}(-\rho, x), \quad (163a)$$

$$K_{332}(\rho, x) = i(\phi_{33}(\rho, x) - \phi_{33}(-\rho, x)). \quad (163b)$$

In order to extract the singular terms we expand  $K_{331}$  and  $K_{332}$  into series as  $\rho \rightarrow \infty$ . Following asymptotic expansions are obtained using MAPLE:

$$K_{331}^{\infty}(\rho, x) = \left\{ c_{10} + \frac{c_{11}}{\rho} + \frac{c_{12}}{\rho^2} + \cdots + \frac{c_{112}}{\rho^{12}} \right\} \exp(-\rho x), \quad (164a)$$

$$K_{332}^{\infty}(\rho, x) = \left\{ c_{20} + \frac{c_{21}}{\rho} + \frac{c_{22}}{\rho^2} + \cdots + \frac{c_{212}}{\rho^{12}} \right\} \exp(-\rho x), \quad (164b)$$

where the leading terms are given by

$$c_{10} = -\frac{1}{2}\eta(\kappa - 1), \quad c_{20} = \frac{1}{2}(\kappa + 1). \quad (165a,b)$$

Subtracting the asymptotic expansions from the integrands in (162), using integration cutoff points for the infinite integrals, evaluating some integrals in closed form and taking the limit as  $x \rightarrow 0$ , after some manipulations (162) is reduced to:

$$k_{33}(0, y, t) = \left\{ -\frac{1}{\pi} \frac{1}{t-y} - \eta \frac{\kappa-1}{\kappa+1} \delta(t-y) + h_{33f}(y, t) \right\}, \quad (166a)$$

$$h_{33f}(y, t) = \frac{2}{\pi(\kappa+1)} \left\{ -c_{11} \ln(A_{331}|t-y|) - c_{21} \frac{\pi}{2} \text{sign}(t-y) + J_{331}(y, t) + J_{332}(y, t) \right\}, \quad (166b)$$

where  $A_{331}$  is an integration cutoff point,  $\delta()$  is the Dirac delta function,

$$\begin{aligned} J_{331}(y, t) &= \int_0^{A_{331}} [K_{331}(\rho, 0) - c_{10}] \cos(\rho(y-t)) d\rho \\ &+ \int_{A_{331}}^{\infty} [K_{331}(\rho, 0) - K_{331}^{\infty}(\rho, 0)] \cos(\rho(y-t)) d\rho \\ &+ \int_{A_{331}}^{\infty} [K_{331}^{\infty}(\rho, 0) - c_{10} - c_{11}/\rho] \cos(\rho(y-t)) d\rho \\ &- c_{11} \left\{ \gamma_0 + \int_0^{A_{331}|t-y|} \frac{\cos(\alpha) - 1}{\alpha} d\alpha \right\}, \end{aligned} \quad (167a)$$

$$\begin{aligned} J_{332}(y, t) &= \int_0^{A_{332}} [K_{332}(\rho, 0) - c_{20}] \sin(\rho(y-t)) d\rho \\ &+ \int_{A_{332}}^{\infty} [K_{332}(\rho, 0) - K_{332}^{\infty}(\rho, 0)] \sin(\rho(y-t)) d\rho \\ &+ \int_{A_{332}}^{\infty} [K_{332}^{\infty}(\rho, 0) - c_{20} - c_{21}/\rho] \sin(\rho(y-t)) d\rho, \end{aligned} \quad (167b)$$

$A_{332}$  is another integration cutoff point and  $\gamma_0$  is the Euler number, [10]. Second integrals in (167a) and (167b) will be neglected in numerical computation for sufficiently large values of  $A_{331}$  and  $A_{332}$ . Third integrals in (167) are evaluated in closed form. The expressions used in the evaluation of these integrals are given in Appendix A. Also, note that first two terms in (166a) are the singular terms and they are the standard expressions obtained for homogeneous materials. (see, for example, equation (22c) in Dag and Erdogan [8]).

*The contact problem for decreasing stiffness ( $\gamma < 0$ )*

Consider the sliding contact problem for a graded medium without a surface crack and remote loading  $\epsilon_0$ . The half-plane is thus subjected to a pair of unbalanced resultant forces  $P$  and  $\eta P$ . The integral equation for this problem can be written as follows:

$$\int_a^b k_{33}(0, y, t) f_3(t) dt = \frac{4\mu_0}{\kappa + 1} f(y), \quad a < y < b, \quad (168)$$

If we now consider (166) and (167) and expand  $K_{331}(\rho, 0)$  and  $K_{332}(\rho, 0)$  into series as  $\rho$  tends to zero, we find the following expansions:

$$K_{331}^\infty(\rho, 0) = b_2 \rho^2 + b_4 \rho^4 + b_6 \rho^6 + O(\rho^8), \quad (169a)$$

$$K_{332}^\infty(\rho, 0) = a_1 \rho + a_3 \rho^3 + a_5 \rho^5 + O(\rho^7), \quad (169b)$$

where,

$$b_2 = \frac{4(2 - \kappa)\eta}{\gamma^2(1 + \kappa)}, \quad (170a)$$

$$b_4 = \frac{-16\eta\{\kappa^2(1 + \text{sign}(\gamma)) - \kappa(7 + 9\text{sign}(\gamma)) + 10 + 16\text{sign}(\gamma)\}}{\gamma^4(\kappa + 1)^2(1 + \text{sign}(\gamma))}, \quad (170b)$$

$$a_1 = \frac{2(\kappa - 1)}{\gamma(\kappa + 1)}, \quad a_3 = \frac{8(k - 3)(\kappa(1 + \text{sign}(\gamma)) - 4)}{\gamma^3(1 + \kappa)^2(1 + \text{sign}(\gamma))}. \quad (170\text{c,d})$$

Observing that  $\text{sign}(\gamma) = 1$  for  $\gamma > 0$ ,  $\text{sign}(\gamma) = -1$  for  $\gamma < 0$  from (169) and (170) it is seen that  $K_{331}(\rho, 0)$  and  $K_{332}(\rho, 0)$  are well behaved near  $\rho = 0$ , for  $\gamma > 0$ . However, for  $\gamma < 0$  coefficients  $b_4$  and  $a_3$  (and possibly that of higher powers of  $\rho$ ) become unbounded and as a result  $k_{33}$  expressed by (166a) also becomes unbounded. Consequently, it is seen that for a graded half plane with an exponentially decaying stiffness the contact problem is not a well-posed problem. Physically, the problem that is analogous to  $\gamma < 0$  case is a homogeneous strip of finite thickness under an unbalanced transverse load  $P$  (in thickness direction) which has no solution (see Ratwani and Erdogan [9], for explanation). Thus for graded half-planes with or without a crack if  $\gamma < 0$  the contact problem has no solution.

#### 4. Singular behavior of the solution

The integral equations of the problem are given by (65) and the kernels of the equations are derived in Section 3. The asymptotic behaviors of the integrands are also examined and singular terms are extracted. Using the expressions given in section 3, integral equations given by (65) can be written as follows:

$$\begin{aligned} \sigma_{yy}(x, 0) \exp(-\gamma x) &= \int_0^d \left[ \frac{1}{\pi} \frac{1}{t-x} + h_{11s}(x, t) + h_{11f}(x, t) \right] f_1(t) dt + \\ &+ \int_a^b \left[ h_{13s}(x, t) + h_{13f}(x, t) \right] f_3(t) dt + E_0 \epsilon_0 = 0, \quad 0 < x < d, \end{aligned} \quad (171\text{a})$$

$$\begin{aligned} \sigma_{xy}(x, 0) \exp(-\gamma x) &= \int_0^d \left[ \frac{1}{\pi} \frac{1}{t-x} + h_{22s}(x, t) + h_{22f}(x, t) \right] f_2(t) dt + \\ &+ \int_a^b \left[ h_{23s}(x, t) + h_{23f}(x, t) \right] f_3(t) dt = 0, \quad 0 < x < d, \end{aligned} \quad (171\text{b})$$

$$\begin{aligned}
& \frac{4\mu_0}{1+\kappa} \frac{\partial}{\partial y} u(0, y) = \int_0^d \left[ h_{31s}(y, t) + h_{31f}(y, t) \right] f_1(t) dt + \\
& + \int_0^d \left[ h_{32s}(y, t) + h_{32f}(y, t) \right] f_2(t) dt - \\
& - \eta \frac{\kappa-1}{\kappa+1} f_3(y) + \int_a^b \left[ -\frac{1}{\pi} \frac{1}{t-y} + h_{33f}(y, t) \right] f_3(t) dt, \quad a < y < b, \tag{171c}
\end{aligned}$$

where expressions for  $h_{ijs}(x^*, t)$  and  $h_{ijf}(x^*, t)$  ( $x^* = x$  for  $i = 1, 2$  and  $x^* = y$  for  $i = 3$ ) are given in Section 3.  $h_{ijs}(x^*, t)$  are the generalized Cauchy kernels (of the order  $1/t$ ) that become unbounded as the arguments  $x^*$  and  $t$  tend to the end point simultaneously.  $h_{ijf}(x^*, t)$  are bounded Fredholm kernels. The singular terms are found to be:

$$\begin{aligned}
\lim_{(x,t) \rightarrow 0} h_{11s}(x, t) &= \lim_{(x,t) \rightarrow 0} h_{22s}(x, t) = \frac{1}{\pi} \left\{ \frac{1}{t+x} + \frac{2t}{(t+x)^2} - \frac{4t^2}{(t+x)^3} \right\}, \\
0 < (t, x) &< d, \tag{172a}
\end{aligned}$$

$$\lim_{(x,t) \rightarrow 0} h_{13s}(x, t) = \frac{1}{\pi} \left\{ \frac{2xt^2}{(t^2+x^2)^2} - \eta \frac{2t^3}{(t^2+x^2)} \right\}, \quad a < t < b, 0 < x < d, \tag{172b}$$

$$\lim_{(x,t) \rightarrow 0} h_{23s}(x, t) = \frac{1}{\pi} \left\{ \eta \frac{2xt^2}{(t^2+x^2)^2} - \frac{2tx^2}{(t^2+x^2)^2} \right\}, \quad a < t < b, 0 < x < d, \tag{172c}$$

$$\lim_{(y,t) \rightarrow 0} h_{31s}(y, t) = -\frac{1}{\pi} \frac{4t^2 y}{(t^2+y^2)^2}, \quad 0 < t < d, \quad a < y < b, \tag{172d}$$

$$\lim_{(y,t) \rightarrow 0} h_{32s}(y, t) = \frac{1}{\pi} \frac{4t^3}{(t^2+y^2)^2}, \quad 0 < t < d, \quad a < y < b, \tag{172e}$$

Note that the singular terms in the integral equations, i.e., the Cauchy singularities and generalized Cauchy kernels given by (172) are also obtained for the crack/contact problem in a homogeneous half-plane. If we compare (172) and equations (23a,e) which are given in [8], we observe that the singular terms are identical except for the sign changes for some terms, which are due to the different coordinate axes used in [8] and in

this report. It may then be concluded that the singular behavior of the unknown functions for the graded and homogeneous materials are identical. Hence, for the graded materials, the singular behavior of the unknown functions is independent of the material nonhomogeneity constants  $\gamma$  and  $\mu_0$  and depend on the friction coefficient  $\eta$  and the surface value of the Poisson's ratio (through elastic constant  $\kappa$ ) only. The details of the function-theoretic analysis to determine the singular behavior of the unknowns are given in [8]. Here we summarize the results, for the two cases  $a > 0$  and  $a = 0$ .

$$a > 0$$

In this case the kernels (172b-e) are bounded in their corresponding closed intervals and would not contribute to the singularities of the functions  $f_1$ ,  $f_2$  and  $f_3$ . We express  $f_i$  as,

$$f_1(x) = x^{\theta_1}(d-x)^{\lambda_1} F_1(x), \quad 0 < x < d, \quad (173a)$$

$$f_2(x) = x^{\theta_2}(d-x)^{\lambda_2} F_2(x), \quad 0 < x < d, \quad (173b)$$

$$f_3(y) = (y-a)^\omega(b-y)^\beta F_3(y), \quad a < y < b. \quad (173c)$$

The function-theoretic analysis to determine the exponents is described in [8], and following equations are obtained,

$$\theta_1 = 0, \quad \theta_2 = 0, \quad (174a)$$

$$\cot(\pi\lambda_1) = 0, \quad \cot(\pi\lambda_2) = 0, \quad (\lambda_1 = \lambda_2 = -0.5), \quad (174b)$$

$$\cot(\pi\omega) = \eta \frac{\kappa-1}{\kappa+1}, \quad \cot(\pi\beta) = -\eta \frac{\kappa-1}{\kappa+1}, \quad (174c,d)$$

where acceptable roots are  $\lambda_1 = -0.5$ ,  $\lambda_2 = -0.5$ ,  $\Re(\omega) < 0$  if  $a$  is known and is a sharp corner,  $\Re(\omega) > 0$ , if  $a$  is unknown and is a point of smooth contact. Similarly

$\Re(\beta) < 0$  if  $b$  is a known sharp corner and  $\Re(\beta) > 0$  if  $b$  is unknown and the contact is smooth.

$$a = 0$$

In this case all kernels  $h_{ijs}(x^*, t)$  become unbounded as  $x^* \rightarrow 0$  and  $t \rightarrow 0$  simultaneously and contribute to the singularity of the unknown functions. Again, we express the unknown functions as follows,

$$f_1(x) = x^\alpha (d - x)^{\lambda_1} G_1(x), \quad 0 < x < d, \quad (175a)$$

$$f_2(x) = x^\alpha (d - x)^{\lambda_2} G_2(x), \quad 0 < x < d, \quad (175b)$$

$$f_3(y) = y^\alpha (b - y)^\beta G_3(x), \quad 0 < y < b, \quad (175c)$$

The function-theoretic analysis carried out in [8] shows that

$$\cot(\pi\lambda_1) = 0, \quad \cot(\pi\lambda_2) = 0, \quad (\lambda_1 = \lambda_2 = -0.5), \quad (176a,b)$$

$$\cot(\pi\beta) = -\eta \frac{\kappa - 1}{\kappa + 1}. \quad (176c)$$

As shown in [8] equation (56) and (57), the eigenvalue  $\alpha$  and the expressions relating  $G_2(0)$  and  $G_3(0)$  are given by

$$\begin{aligned} & \frac{2\alpha^2 + 4\alpha + 1 - \cos(\pi\alpha)}{(\kappa + 1)\sin^2(\pi\alpha)} \times \\ & \times (\eta(4\alpha^2 + 10\alpha + 5 + (\kappa - 1)\cos(\pi\alpha) + \kappa(2\alpha + 3)) + (\kappa + 1)\sin(\pi\alpha)) = 0, \end{aligned} \quad (177a)$$

$$G_1(0)\sqrt{d} = -\left\{ \frac{\eta(\alpha + 2)\cos(\pi\alpha/2) + (\alpha + 1)\sin(\pi\alpha/2)}{2\alpha^2 + 4\alpha + 1 - \cos(\pi\alpha)} \right\} G_3(0)b^\beta, \quad (177b)$$

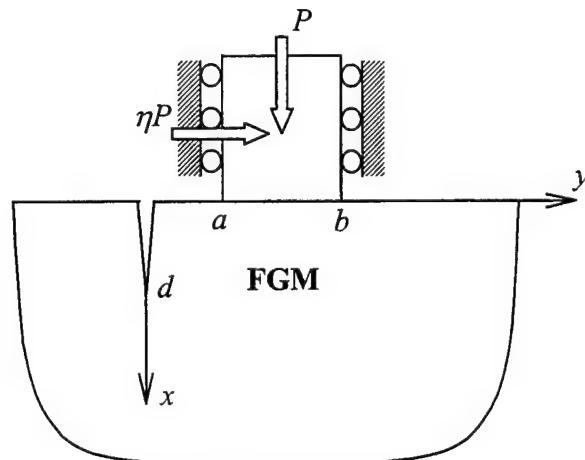
$$G_2(0)\sqrt{d} = -\left\{ \frac{\eta(\alpha + 1)\sin(\pi\alpha/2) - \alpha\cos(\pi\alpha/2)}{2\alpha^2 + 4\alpha + 1 - \cos(\pi\alpha)} \right\} G_3(0)b^\beta. \quad (177c)$$

From the symmetrically loaded half plane ( $y > 0$ ) analogy it is known that  $\alpha = 0$  for  $\eta = 0$  and at  $x = 0, y = 0$  the stress state is bounded. In this case function-theoretic analysis shows that in order not to have a logarithmic singularity in the integral equations following condition must be satisfied,

$$G_2(0)\sqrt{d} = \frac{1}{4}G_3(0)b^\beta. \quad (178)$$

## 5. Numerical solution of the integral equations

In this section, we will develop a numerical solution method for the case of a flat stamp as shown in Figure 2. We first normalize the intervals in (171) by defining



**Figure 2:** The geometry of the crack/contact problem for a flat stamp.

$$t = \frac{d}{2}r + \frac{d}{2}, \quad t = \frac{d}{2}r + \frac{d}{2}, \quad t = \frac{b-a}{2}r + \frac{b+a}{2} \quad (179a,b,c)$$

in integrals involving  $f_1(t)$ ,  $f_2(t)$  and  $f_3(t)$ , respectively. Then we define the normalized unknowns of the problem as follows:

$$\phi_1(r) = \frac{f_1 \left( \frac{d}{2}r + \frac{d}{2} \right)}{P/(b-a)}, \quad -1 < r < 1, \quad (180a)$$

$$\phi_2(r) = \frac{f_2 \left( \frac{d}{2}r + \frac{d}{2} \right)}{P/(b-a)}, \quad -1 < r < 1, \quad (180b)$$

$$\phi_3(r) = \frac{f_3 \left( \frac{b-a}{2}r + \frac{b+a}{2} \right)}{P/(b-a)}, \quad -1 < r < 1. \quad (180c)$$

The intervals  $(0, d)$  and  $(a, b)$  are also normalized by defining,

$$x = \frac{d}{2}s_1 + \frac{d}{2}, \quad \text{for eqn. (171a),} \quad (181a)$$

$$x = \frac{d}{2}s_2 + \frac{d}{2}, \quad \text{for eqn. (171b),} \quad (181b)$$

$$y = \frac{b-a}{2}s_3 + \frac{b+a}{2}, \quad \text{for eqn. (171c).} \quad (181c)$$

Using (180) and (181), integral equations (171) and equilibrium condition (7) can be written as

$$\begin{aligned} \frac{1}{\pi} \int_{-1}^1 \frac{\phi_1(r)}{r-s_1} dr + \int_{-1}^1 H_{11}(s_1, r)\phi_1(r)dr + \int_{-1}^1 H_{13}(s_1, r)\phi_3(r)dr + \\ + \frac{E_0\epsilon_0}{P/(b-a)} = 0, \quad -1 < s_1 < 1, \end{aligned} \quad (182a)$$

$$\begin{aligned} \frac{1}{\pi} \int_{-1}^1 \frac{\phi_2(r)}{r-s_2} dr + \int_{-1}^1 H_{22}(s_2, r)\phi_2(r)dr + \int_{-1}^1 H_{23}(s_2, r)\phi_3(r)dr = \\ = 0, \quad -1 < s_2 < 1, \end{aligned} \quad (182b)$$

$$\begin{aligned} \int_{-1}^1 H_{31}(s_3, r)\phi_1(r)dr + \int_{-1}^1 H_{32}(s_3, r)\phi_2(r)dr - \eta \frac{\kappa-1}{\kappa+1} \phi_3(s_3) + \\ + \int_{-1}^1 H_{33}(s_3, r)\phi_3(r)dr = 0, \quad -1 < s_3 < 1, \end{aligned} \quad (182c)$$

$$\int_{-1}^1 \phi_3(s_3) ds_3 = -2, \quad (182d)$$

where  $H_{ij}(s_i, r)$  are given in Appendix C. Unknown function  $\phi_i(r)$ , ( $i = 1, 2, 3$ ) can now be expressed in the following form,

$$\phi_1(r) = w_1(r) \sum_{n=0}^{\infty} A_{1n} P_n^{(-1/2, \alpha_1)}(r), \quad w_1(r) = (1-r)^{-1/2} (1+r)^{\alpha_1}, \quad (183a)$$

$$\phi_2(r) = w_2(r) \sum_{n=0}^{\infty} A_{2n} P_n^{(-1/2, \alpha_1)}(r), \quad w_2(r) = (1-r)^{-1/2} (1+r)^{\alpha_1}, \quad (183b)$$

$$\phi_3(r) = w_3(r) \sum_{n=0}^{\infty} A_{3n} P_n^{(\beta, \alpha_2)}(r), \quad w_3(r) = (1-r)^{\beta} (1+r)^{\alpha_2}, \quad (183c)$$

where for  $a = 0$ ,  $\alpha_1 = \alpha_2 = \alpha$  and for  $a > 0$   $\alpha_1 = 0$ ,  $\alpha_2 = \omega$ . Substituting (183c) into (182d),  $A_{30}$  is obtained as

$$A_{30} = -2/\theta_0, \quad \theta_0 = \frac{2^{\beta+\alpha_2+1} \Gamma(\beta+1) \Gamma(\alpha_2+1)}{\Gamma(\beta+\alpha_2+2)}. \quad (184a,b)$$

Now, substituting (183) into (182), regularizing the singular parts of the equations using the expressions given in Appendix D in [8] and truncating the infinite series at  $N$ , following system of linear algebraic equations is obtained:

$$\sum_{n=0}^N m_{11n}(s_1) A_{1n} + \sum_{n=1}^N m_{13n}(s_1) A_{3n} = -\frac{E_0 \epsilon_0}{P/(b-a)} - m_{130}(s_1) A_{30}, \\ -1 < s_1 < 1, \quad (185a)$$

$$\sum_{n=0}^N m_{22n}(s_2) A_{2n} + \sum_{n=1}^N m_{23n}(s_2) A_{3n} = -m_{230}(s_2) A_{30}, \quad -1 < s_2 < 1, \quad (185b)$$

$$\sum_{n=0}^N m_{31n}(s_3) A_{1n} + \sum_{n=0}^N m_{32n}(s_3) A_{2n} + \sum_{n=1}^N m_{33n}(s_3) A_{3n} = -m_{330}(s_3) A_{30}, \\ -1 < s_3 < 1. \quad (185c)$$

The expressions for  $m_{ijn}(s_i)$ , ( $i, j = 1, 2, 3$ ) are given in Appendix C. Note that if  $a = 0$ , compatibility conditions expressed by (177) and (178) must also be considered. Substituting (183) in (177) and (178) we obtain,

$$\eta > 0$$

$$\begin{aligned} \sum_{n=0}^N A_{1n} P_n^{(-1/2, \alpha)}(-1) - f_1(\alpha) \left(\frac{b}{d}\right)^{-\alpha} 2^{\beta+1/2} \sum_{n=1}^N A_{3n} P_n^{(\beta, \alpha)}(-1) = \\ = f_1(\alpha) \left(\frac{b}{d}\right)^{-\alpha} 2^{\beta+1/2} A_{30}, \end{aligned} \quad (186a)$$

$$\begin{aligned} \sum_{n=0}^N A_{2n} P_n^{(-1/2, \alpha)}(-1) - f_2(\alpha) \left(\frac{b}{d}\right)^{-\alpha} 2^{\beta+1/2} \sum_{n=1}^N A_{3n} P_n^{(\beta, \alpha)}(-1) = \\ = f_2(\alpha) \left(\frac{b}{d}\right)^{-\alpha} 2^{\beta+1/2} A_{30}, \end{aligned} \quad (186b)$$

$$f_1(\alpha) = - \frac{\eta(\alpha+2)\cos(\pi\alpha/2) + (\alpha+1)\sin(\pi\alpha/2)}{2\alpha^2 + 4\alpha + 1 - \cos(\pi\alpha)}, \quad (186c)$$

$$f_2(\alpha) = - \frac{\eta(\alpha+1)\sin(\pi\alpha/2) - \alpha\cos(\pi\alpha/2)}{2\alpha^2 + 4\alpha + 1 - \cos(\pi\alpha/2)}, \quad (186d)$$

$$\eta = 0$$

$$\sum_{n=0}^N A_{2n} P_n^{(-1/2, 0)}(-1) - \frac{1}{4} 2^{\beta+1/2} \sum_{n=1}^N A_{3n} P_n^{(\beta, 0)}(-1) = \frac{1}{4} 2^{\beta+1/2} A_{30}, \quad (187)$$

Equations (185) can be solved using the collocation technique. For  $a > 0$ , the number of unknowns is  $(3N + 2)$ . Roots of the Chebyshev polynomials are used as the collocation points as follows:

$$s_{1i} = \cos\left(\frac{\pi(2i-1)}{2(N+1)}\right), \quad i = 1, \dots, N+1, \quad (188a)$$

$$s_{2i} = \cos\left(\frac{\pi(2i-1)}{2(N+1)}\right), \quad i = 1, \dots, N+1, \quad (188b)$$

$$s_{3i} = \cos\left(\frac{\pi(2i-1)}{2N}\right), \quad i = 1, \dots, N. \quad (188c)$$

In the numerical solution for  $a = 0$ , the equations (186) and (187) are also considered. After solving equations (185) for  $A_{in}$ , ( $i = 1, 2, 3$ ) the contact stresses  $\sigma_{xx}(0, y)$  and  $\sigma_{xy}(0, y)$  and stress intensity factors at the crack tip  $(d, 0)$  may be evaluated by using the results. The stress intensity factors are defined by and calculated from

$$\begin{aligned} k_I &= \lim_{x \rightarrow d+0} \sqrt{2(x-d)} \sigma_{yy}(x, 0) = \\ &= - \lim_{x \rightarrow d-0} \frac{2\mu(x)}{\kappa+1} \sqrt{2(d-x)} \frac{\partial}{\partial} (v(x, 0^+) - v(x, 0^-)), \end{aligned} \quad (189a)$$

$$\begin{aligned} k_{II} &= \lim_{x \rightarrow d+0} \sqrt{2(x-d)} \sigma_{xy}(x, 0) = \\ &= - \lim_{x \rightarrow d-0} \frac{2\mu(x)}{\kappa+1} \sqrt{2(d-x)} \frac{\partial}{\partial x} (u(x, 0^+) - u(x, 0^-)). \end{aligned} \quad (189b)$$

Using (189), the normalized stress intensity factors and the normal component of the contact stress may be expressed as

$$\frac{k_I \sqrt{d}}{P} = - \exp(\gamma d) 2^{\alpha_1} \frac{d}{b-a} \sum_{n=0}^N A_{1n} P_n^{(-1/2, \alpha_1)}(1), \quad (190a)$$

$$\frac{k_{II} \sqrt{d}}{P} = - \exp(\gamma d) 2^{\alpha_1} \frac{d}{b-a} \sum_{n=0}^N A_{2n} P_n^{(-1/2, \alpha_1)}(1), \quad (190b)$$

$$\frac{\sigma_{xx}\left(0, \frac{b-a}{2}s_3 + \frac{b+a}{2}\right)}{P/(b-a)} = (1-s_3)^\beta (1+s_3)^{\alpha_2} \sum_{n=0}^N A_{3n} P_n^{(\beta, \alpha_2)}(s_3). \quad (191)$$

## 6. Results and discussion

The calculated results in this report consist of the normal and in-plane components of the stresses on the surface of the graded half plane in the absence of a crack, contact stresses ( $\sigma_{xx}(0, y)$ ,  $\sigma_{xy}(0, y)$ ,  $a < y < b$ ) and the stress intensity factors ( $k_I$ ,  $k_{II}$ ). First we give some results, showing the surface stresses in a graded medium in the absence of a crack and loaded by a sliding flat stamp. These results are shown in Figures 3-8. Contact stresses in the absence of a crack are calculated by solving the integral equation given by (168) and also considering the equilibrium condition given by (7). As shown in section 3.7, the contact problem with or without a crack has no solution for  $\gamma < 0$ . Hence, in Figures 3-8 results are given for positive values of the nonhomogeneity parameter  $\gamma$ . Figures 3 and 4 show that, for  $\eta = 0$  both  $\sigma_{xx}(0, y)$  and  $\sigma_{yy}(0, y)$  are symmetric and they have square-root singularities at  $y = a$  and  $y = b$ .  $\sigma_{yy}(0, y)$  vanishes outside the contact area for  $\gamma = 0$ , and as  $\gamma$  increases it becomes tensile at both ends of the contact region. Figures 5 and 6 show the results for  $\eta = 0.4$ . It is seen that there is a greater stress concentration near the trailing end of the stamp,  $y = a$  and  $|\omega| > |\beta|$ ,  $\omega$  and  $\beta$  being the singularities at  $y = a$  and  $y = b$  respectively. The important conclusion one may draw from Figure 6 is that at the trailing end of the stamp the in-plane component of the stress  $\sigma_{yy}(0, y)$  is unbounded, tensile and discontinuous and has a singularity of the order  $(a - y)^\omega$ , where  $-\omega > 1/2$ . This implies that  $y = a$  is a likely location of surface crack initiation. Similar results are also shown in Figures 7 and 8 for  $\eta = 0.8$ .

In Figures 9-16, stress intensity factors  $k_I$  and  $k_{II}$  are shown as functions of the relative stamp size  $b/d$  for  $a = 0$ ,  $\nu = 0.25$  and for various values of  $\gamma$  and  $\eta$ . In these figures the stress intensity factors are normalized with respect to  $P/\sqrt{d}$  also the nonhomogeneity parameter is used in normalized form  $\gamma d$ . The circles in these figures are the results obtained from the solution of the homogeneous half-plane problem as

described in Dag and Erdogan [8]. It is shown in Figures 9 and 10 and in all the results presented in this study that, the results obtained from the solution of the graded half plane problem by letting  $\gamma d = 0.0001$  and those obtained from the solution of the homogeneous half plane problem are in very good agreement. That is, for all intents and purposes these two sets of results are identical. Figures 9 and 10 show that for  $a = 0$  and  $\eta = 0$ , i.e., for the case of normal indentation, mode I stress intensity factors are negative and mode II stress intensity factors are positive for all values of the nonhomogeneity constant  $\gamma d$ . Mode I stress intensity factors increase and mode II stress intensity factors decrease as  $\gamma d$  increases. Since  $k_I$  is less than zero crack closure occurs and there is contact between the crack faces. But the results can still be applicable and useful in superposition with an uncoupled solution resulting, for example, from remote strain loading  $\epsilon_{yy}(x, \mp\infty)$ , [11], [12], provided the resultant  $k_I$  is positive. Otherwise, the problem needs to be formulated by taking into account the crack closure and determining the closure distance from the condition of  $k_I = 0$ . Figures 11 and 12 show the modes I and II stress intensity factors for  $\eta = 0.2$  and for  $a = 0$ . It is seen that, mode I stress intensity factors increase as the friction coefficient increases but they are still negative for this value of  $\eta$ . Comparison of Figures 10 and 12 shows that mode II stress intensity factors decrease as  $\eta$  decreases. It is also seen that, the effect of the nonhomogeneity parameter on the stress intensity factors is quite significant. Again, the results for  $\gamma d = 0.0001$  are in exact agreement with the results obtained from the homogeneous formulation. In Figures 13-14 and 15-16 modes I and II stress intensity factors are given for  $\eta = 0.4$  and  $\eta = 0.8$ , respectively. It can be observed that gradually  $k_I$  becomes positive and  $k_{II}$  becomes negative as the the tangential force increases. The contact stress distribution is shown in Figures 17-20 for  $a = 0$ . In this case the stress singularities  $\alpha$  and  $\beta$  at the end points  $a = 0$  and  $b$  are given by (56) and (174d), respectively. Figure 17 shows that for  $\eta = 0$  there is no singularity at the trailing end  $a = 0$ . In Figures 18 and

19, contact stress distributions for  $\eta = 0.2$  and  $\eta = 0.4$  are given. For these relatively small values of  $\eta$  the stress singularities at  $b$  are greater than those at  $a = 0$  (i.e.,  $-\beta > -\alpha$ ), hence the skewed distribution in Figures 18 and 19. On the other hand for relatively large values of  $\eta$ ,  $|\alpha| > |\beta|$  the trend is reversed and there is a greater stress concentration near the end  $a = 0$  (see Figure 20).

Figures 21-28 show the modes I and II stress intensity factors as functions of  $a/d$  for a constant relative contact area length  $(b - a)/d = 0.1$ . These figures also show that, the limiting cases of  $\gamma d = 0.0001$ , are in very good agreement with the results obtained from the solution of the homogeneous half plane problem. As seen in Figure 21, for  $\gamma d = 0.0001$  mode I stress intensity factors are negative for all values of  $a/d$ , which would lead to crack closure. It can also be seen that mode I stress intensity factors in a graded medium are larger than those for the homogenous medium and for some values of  $\gamma d$  and  $a/d$ , mode I stress intensity factors become positive. Figure 22 shows that mode II stress intensity factors are positive for all values of  $a/d$  in a homogeneous medium, and they decrease gradually as the nonhomogeneity parameter  $\gamma d$  increases. Figures 23 and 24 show the results for  $\eta = 0.2$ . As the coefficient of friction, hence the tangential force increases mode I stress intensity factors increase and mode II stress intensity factors decrease. The results for  $\eta = 0.4$  and  $\eta = 0.8$  are shown in Figures 25-26 and 27-28 respectively. Figure 27 shows that for  $\eta = 0.8$  mode I stress intensity factors are positive for all values of  $\gamma d$ .

Contact stress distribution for  $(b - a)/d = 0.1$  and  $a/d = 0.4$  are given in Figures 29-32. Figure 29 shows the results for  $\eta = 0$ . Although there is no tangential force and singularities are equal at both ends of the contact area, the stress distribution is not exactly symmetric due to the effect of the surface crack in the graded medium. It can be seen in Figures 30-32 that, as the coefficient of friction increases, singularity at the

leading end i.e.,  $-\beta$  decreases and there is a higher stress intensification at the trailing end.

Another set of results for the stress intensity factors are given in Figures 33-40 for a relatively larger stamp size  $(b - a)/d = 1.0$ . The trends are similar as in Figures 21-28. The contact stress distributions for  $(b - a)/d = 1.0$  are shown in Figures 41-44 for various values of the friction coefficient  $\eta$ .

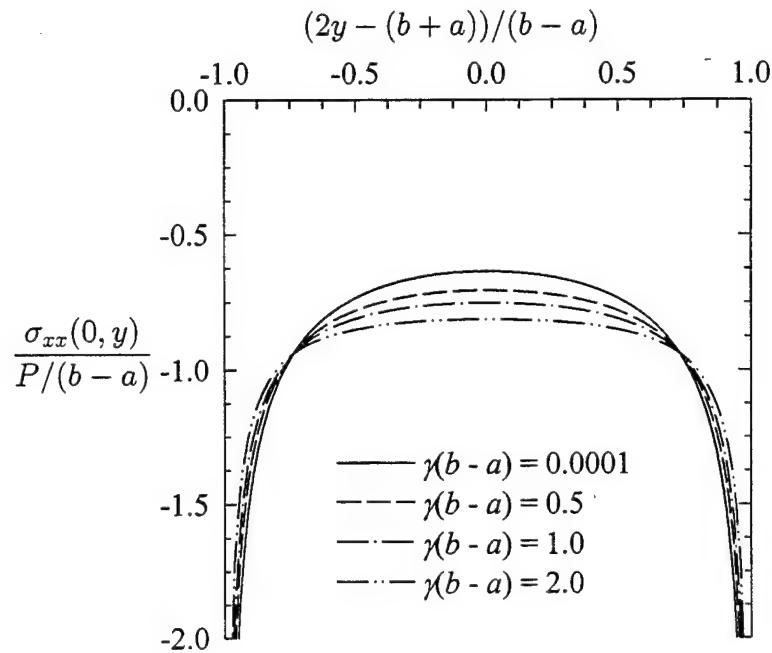
### Some conclusions

1. Analytically the contact problem for a graded half-plane with exponentially decaying stiffness is not a well-posed problem.
2. The trailing end of the sliding rigid stamp with friction is a likely location of surface crack initiation due to greater stress concentration.
3. In the medium containing a surface crack and loaded by a sliding rigid stamp, the mixed mode stress state at the crack tip is such that the cracks tend to be periodic and curve backward.
4. In the coupled crack/contact problems for a graded medium stress singularities  $\alpha$ ,  $\beta$  and  $\omega$  are independent of the material nonhomogeneity constants  $\gamma$  and  $\mu_0$  and depend on the friction coefficient  $\eta$  and the surface value of the Poisson's ratio (through the elastic constant  $\kappa$ ) only.

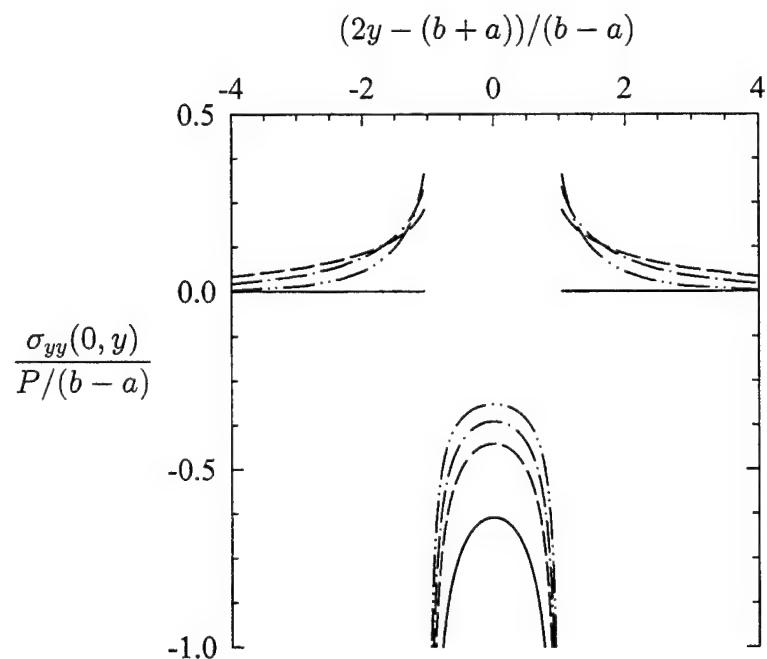
### REFERENCES

- [1] H. Hertz, "Über die Berührung fester elastischer Körper", *J. Reine und Angewandte Mathematik*, 92 (1882) 156-171.
- [2] K.L. Johnson, *Contact Mechanics*, Cambridge University Press (1985)

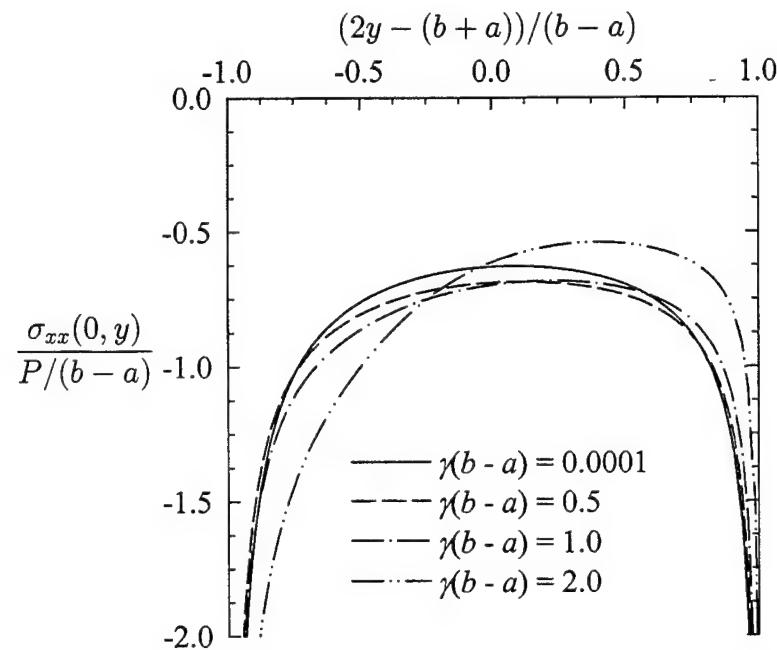
- [3] I. Bakirtas, "The problem of a rigid punch on a nonhomogeneous elastic half space", *International Journal of Engineering Science*, 18 (1980) 597-610.
- [4] A.E. Giannakopoulos and S. Suresh, "Indentation of solids with gradients in elastic properties: Part II Axisymmetric indenters", *International Journal of Solids and Structures*, 34 (1997) 2393-2428.
- [5] S. Suresh, A.E. Giannakopoulos and J. Alcala, "Spherical indentation of compositionally graded materials: Theory and experiments", *Acta Materialia*, 45 (1997) 1307-1321.
- [6] M.A. Guler, "Contact Mechanics of FGM Coatings", Ph.D. Dissertation, Lehigh University, Bethlehem, PA, (2000).
- [7] F. Delale and F. Erdogan, "Crack problem for a nonhomogeneous plane", *ASME Journal of Applied Mechanics*, 50 (1983) 609-614.
- [8] S. Dag and F. Erdogan, "Cracking of a homogeneous half plane due to sliding contact", Technical Report, AFOSR, Grant F49620-98-1-0028, (2001).
- [9] M. Ratwani and F. Erdogan, "On the plane contact problem for a frictionless elastic layer", *International Journal of Solids and Structures*, 9 (1973) 921-936.
- [10] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series and Products*, Academic Press, Boston (1994).
- [11] F. Erdogan and B.H. Wu, "Surface crack problem for a plate with functionally graded properties", *ASME Journal of Applied Mechanics*, 64 (1997) 449-456.
- [12] M. Kasmalkar, "The surface crack problem for a functionally graded coating bonded to a homogeneous layer", Ph.D. Dissertation, Lehigh University, Bethlehem, PA, (1997)



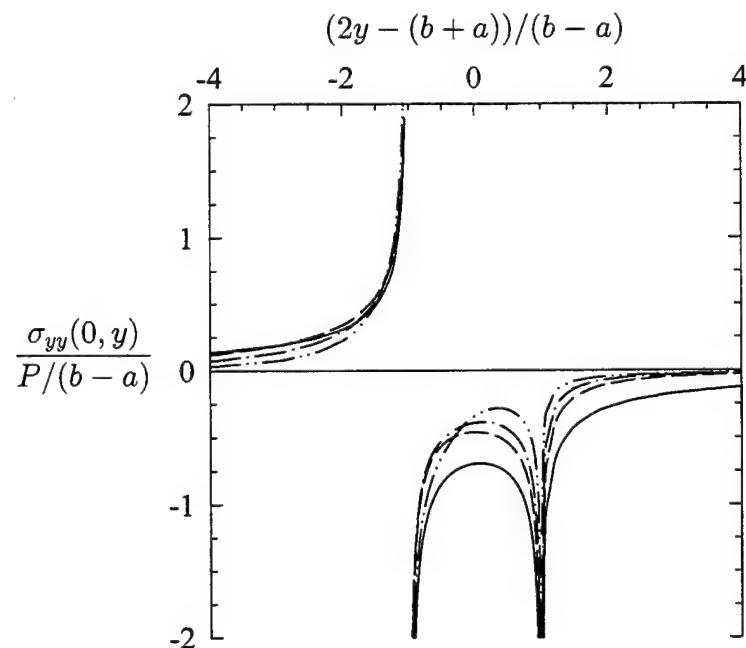
**Figure 3:** The distribution of the contact stress on the surface of the graded medium loaded by a flat stamp as shown in Figure 2,  $d = 0$ ,  $\eta = 0$ ,  $\kappa = 2$ .



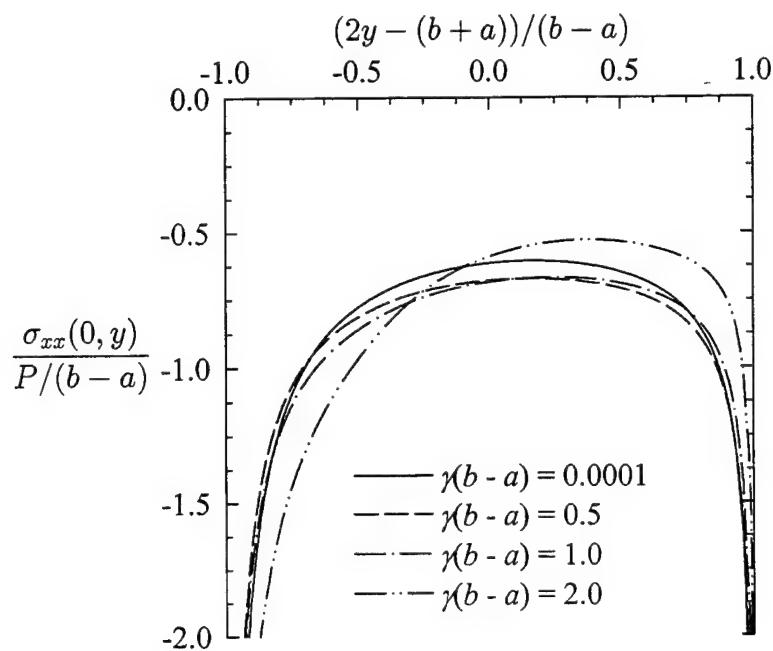
**Figure 4:** The distribution of the in-plane stress on the surface of the graded medium loaded by a flat stamp as shown in Figure 2,  $d = 0$ ,  $\eta = 0$ ,  $\kappa = 2$ .



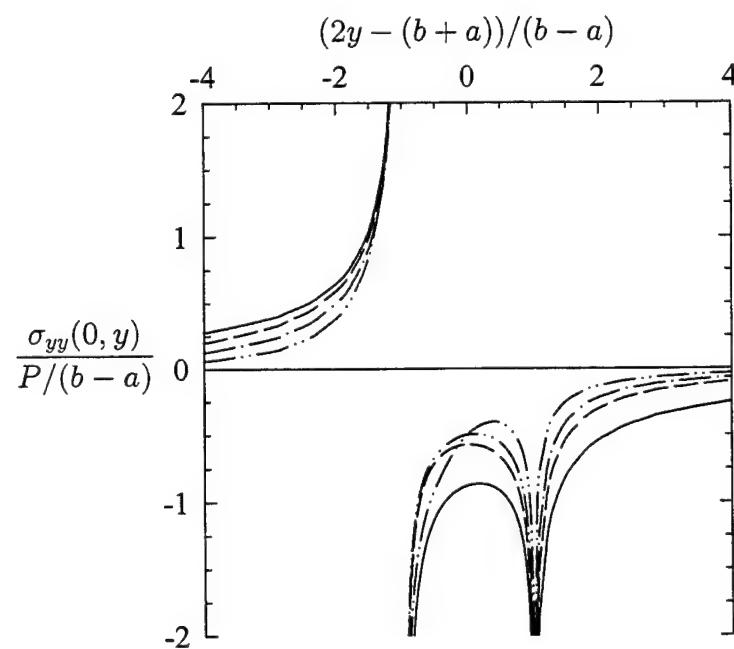
**Figure 5:** The distribution of the contact stress on the surface of the graded medium loaded by a flat stamp as shown in Figure 2 ,  $d = 0$ ,  $\eta = 0.4$ ,  $\kappa = 2$ .



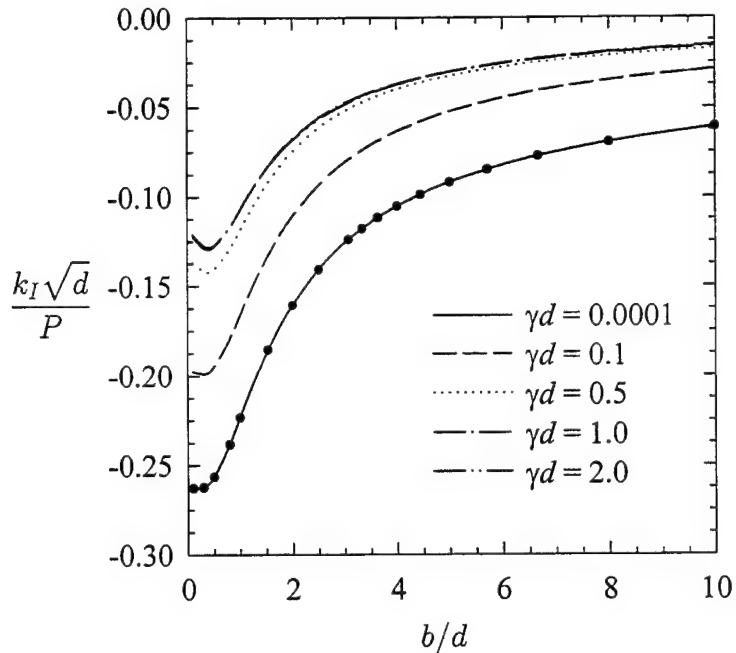
**Figure 6:** The distribution of the in-plane stress on the surface of the graded medium loaded by a flat stamp as shown in Figure 2 ,  $d = 0$ ,  $\eta = 0.4$ ,  $\kappa = 2$ .



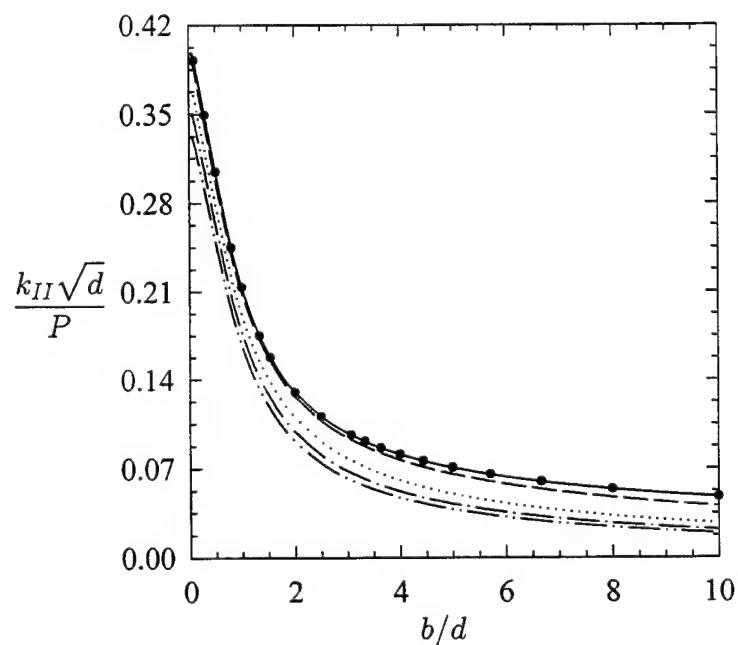
**Figure 7:** The distribution of the contact stress on the surface of the graded medium loaded by a flat stamp as shown in Figure 2 ,  $d = 0$ ,  $\eta = 0.8$ ,  $\kappa = 2$ .



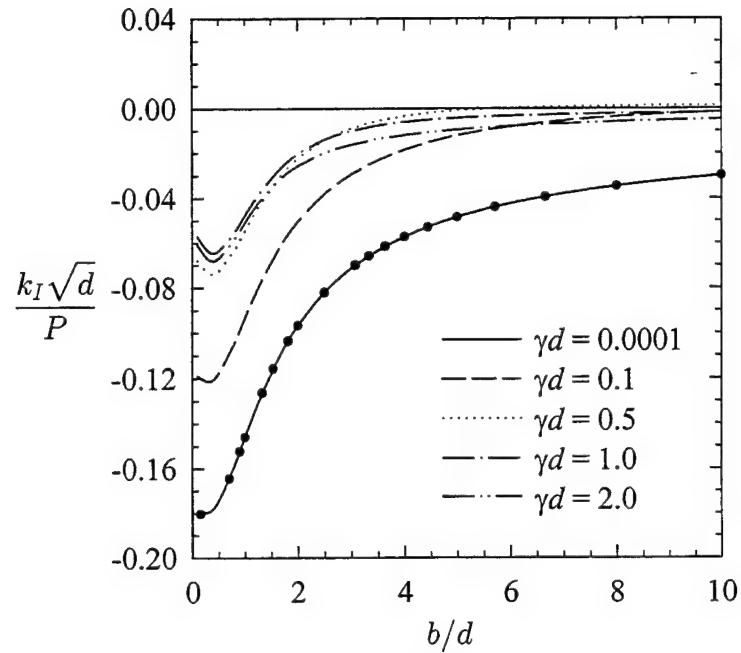
**Figure 8:** The distribution of the in-plane stress on the surface of the graded medium loaded by a flat stamp as shown in Figure 2 ,  $d = 0$ ,  $\eta = 0.8$ ,  $\kappa = 2$ .



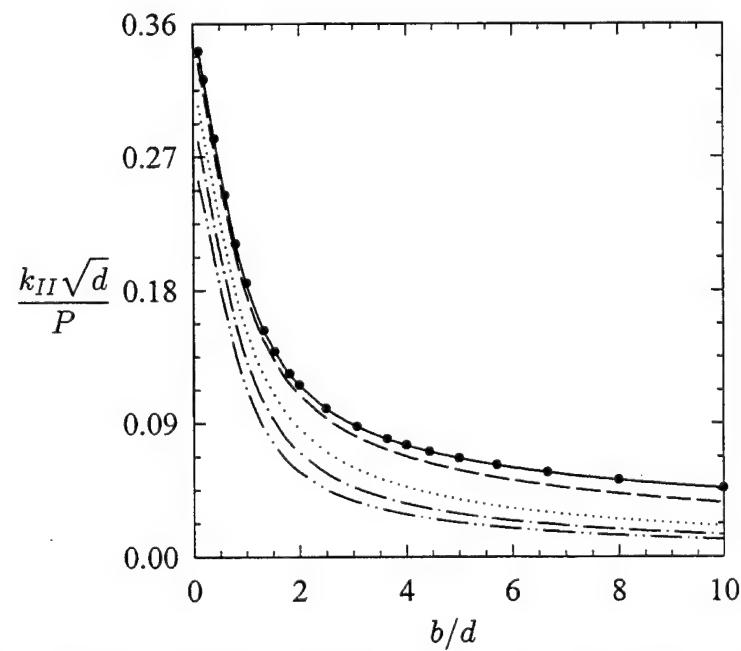
**Figure 9:** Mode I stress intensity factors for an edge crack in a graded half plane indented by a flat stamp as shown in Figure 2,  $a = 0$ ,  $\eta = 0$ ,  $\nu = 0.25$ .



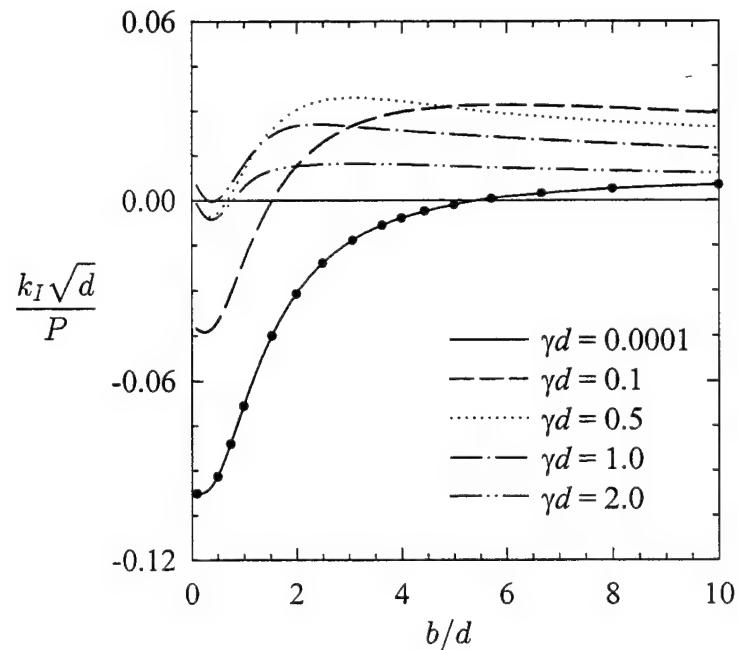
**Figure 10:** Mode II stress intensity factors for an edge crack in a graded half plane indented by a flat stamp as shown in Figure 2,  $a = 0$ ,  $\eta = 0$ ,  $\nu = 0.25$ .



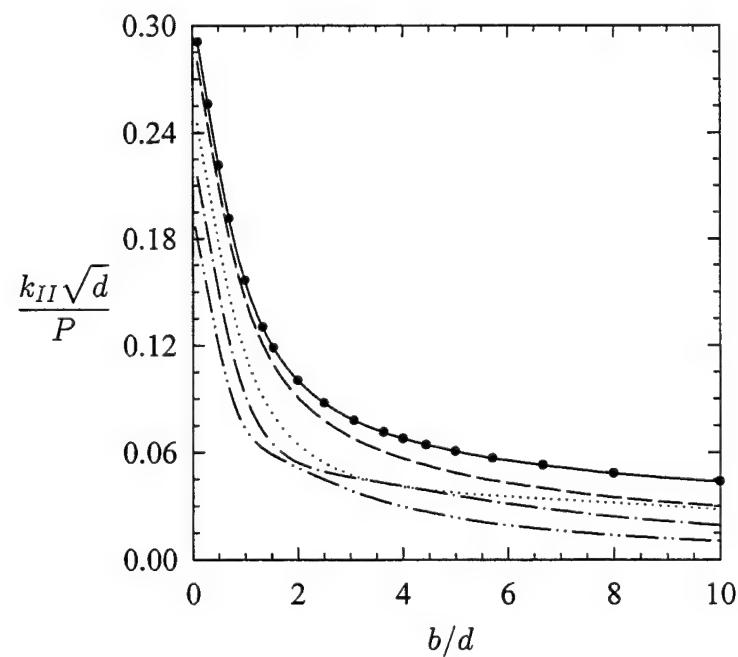
**Figure 11:** Mode I stress intensity factors for an edge crack in a graded half plane indented by a flat stamp as shown in Figure 2,  $a = 0$ ,  $\eta = 0.2$ ,  $\nu = 0.25$ .



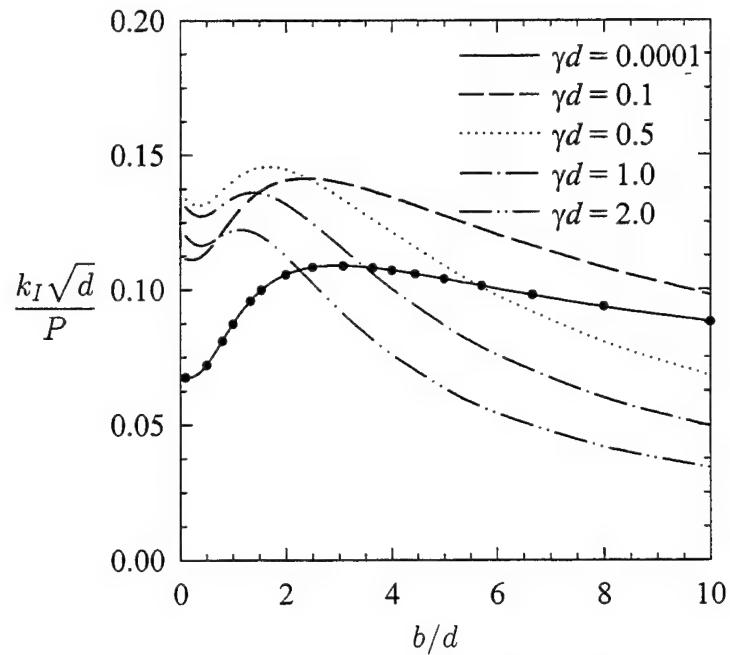
**Figure 12:** Mode II stress intensity factors for an edge crack in a graded half plane indented by a flat stamp as shown in Figure 2,  $a = 0$ ,  $\eta = 0.2$ ,  $\nu = 0.25$ .



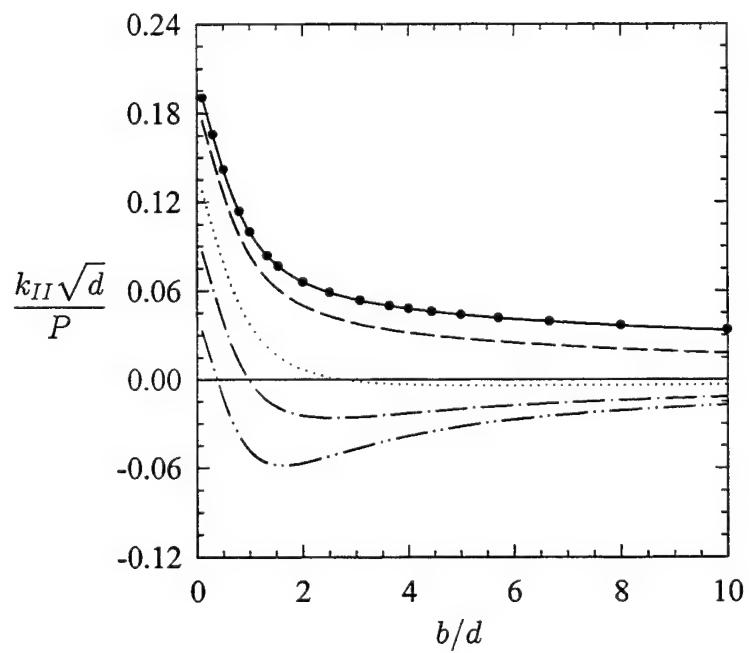
**Figure 13:** Mode I stress intensity factors for an edge crack in a graded half plane indented by a flat stamp as shown in Figure 2,  $a = 0$ ,  $\eta = 0.4$ ,  $\nu = 0.25$ .



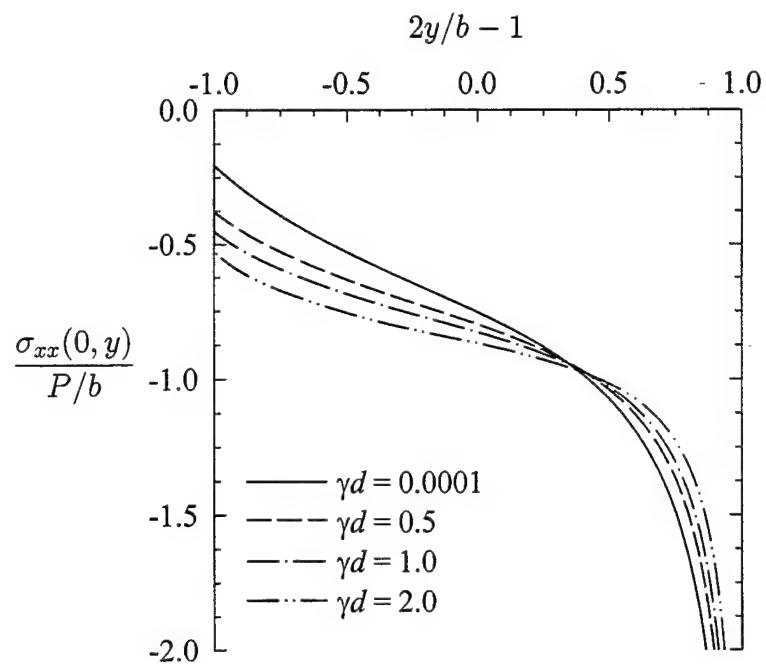
**Figure 14:** Mode II stress intensity factors for an edge crack in a graded half plane indented by a flat stamp as shown in Figure 2,  $a = 0$ ,  $\eta = 0.4$ ,  $\nu = 0.25$ .



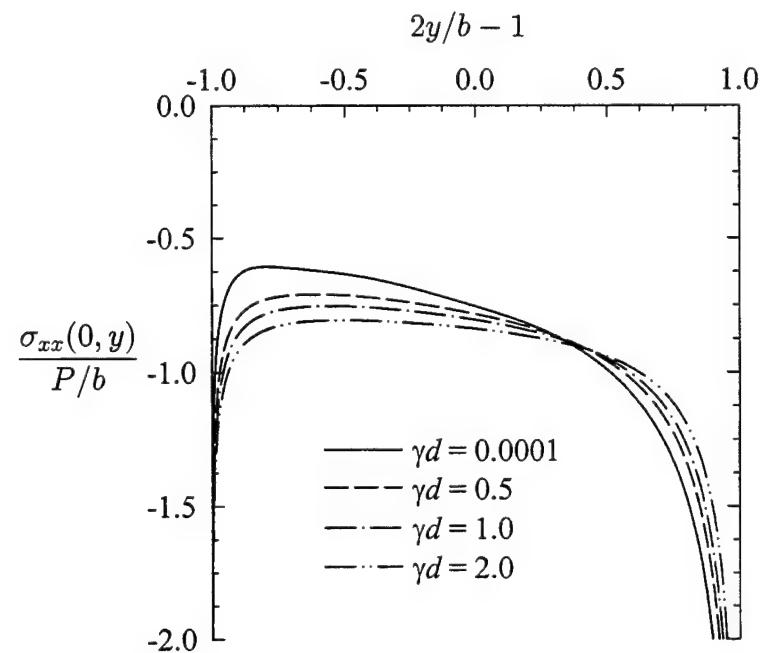
**Figure 15:** Mode I stress intensity factors for an edge crack in a graded half plane indented by a flat stamp as shown in Figure 2,  $a = 0$ ,  $\eta = 0.8$ ,  $\nu = 0.25$ .



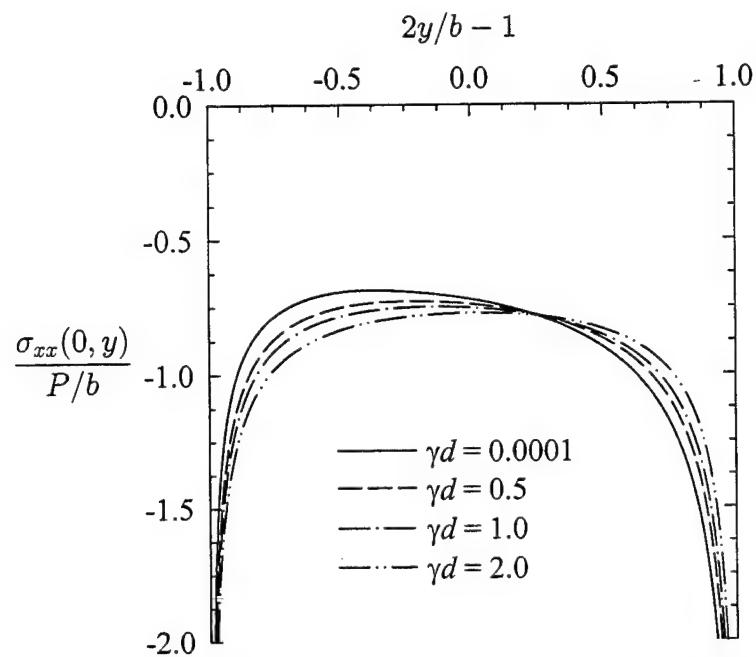
**Figure 16:** Mode II stress intensity factors for an edge crack in a graded half plane indented by a flat stamp as shown in Figure 2,  $a = 0$ ,  $\eta = 0.8$ ,  $\nu = 0.25$ .



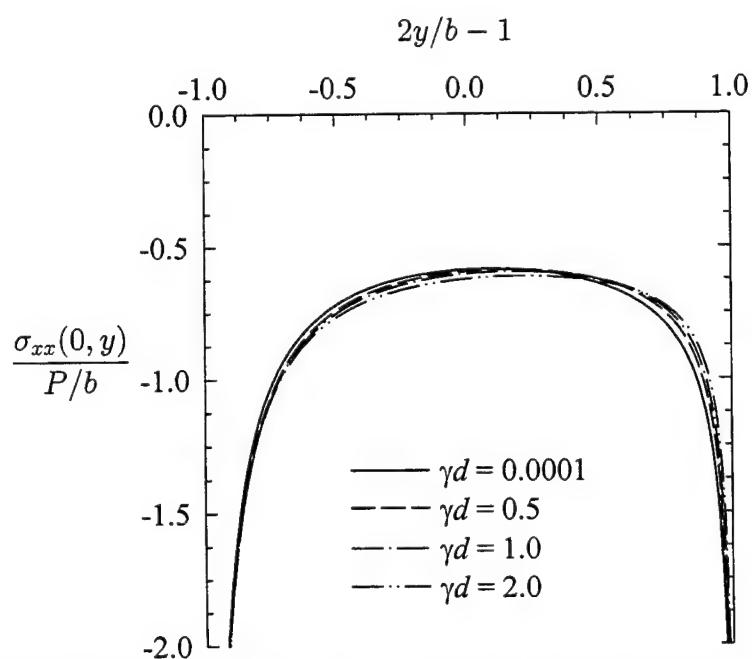
**Figure 17:** Contact stress distribution for a graded half plane with an edge crack and indented by a flat stamp as shown in Figure 2,  $a = 0$ ,  $\eta = 0$ ,  $\nu = 0.25$ ,  $b/d = 0.5$ .



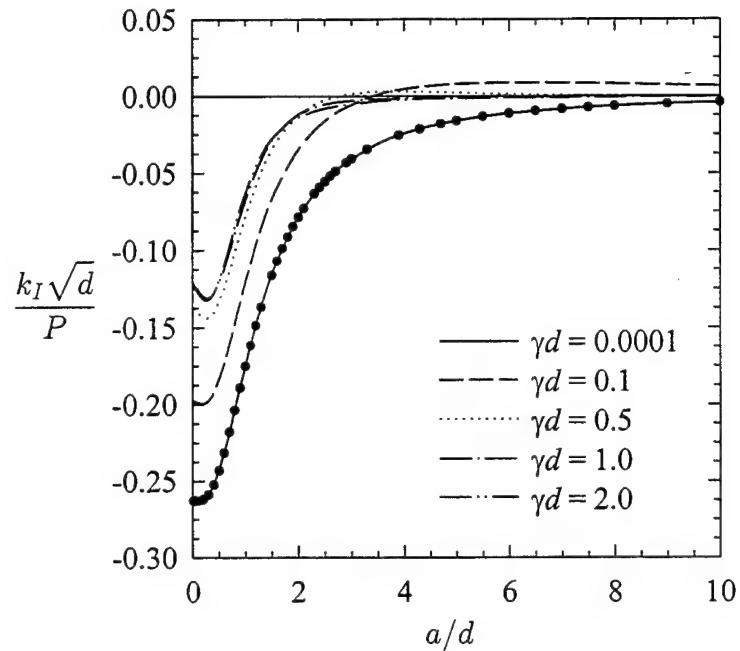
**Figure 18:** Contact stress distribution for a graded half plane with an edge crack and indented by a flat stamp as shown in Figure 2,  $a = 0$ ,  $\eta = 0.2$ ,  $\nu = 0.25$ ,  $b/d = 0.5$ .



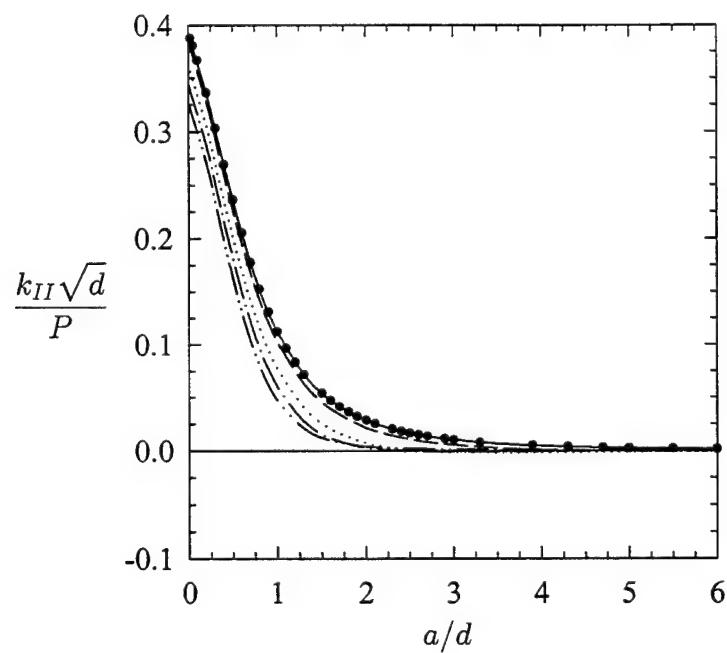
**Figure 19:** Contact stress distribution for a graded half plane with an edge crack and indented by a flat stamp as shown in Figure 2,  $a = 0$ ,  $\eta = 0.4$ ,  $\nu = 0.25$ ,  $b/d = 0.4$ .



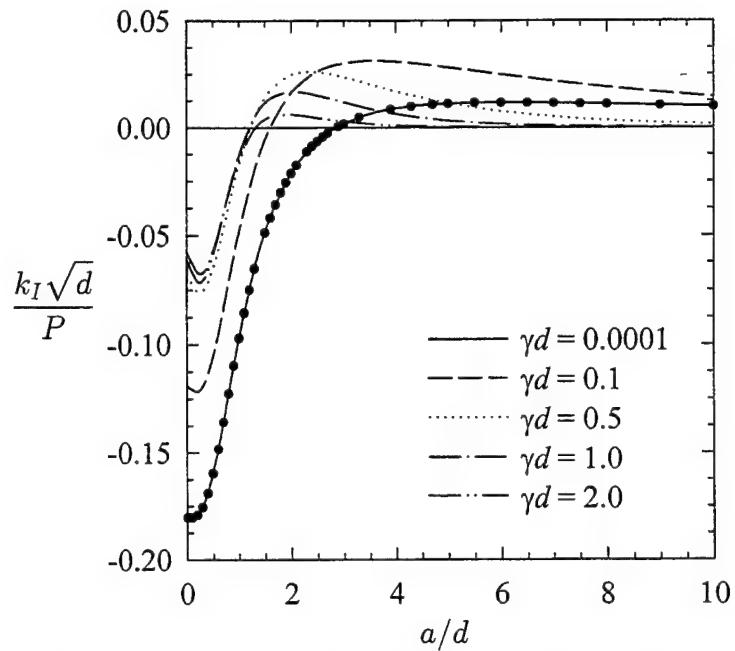
**Figure 20:** Contact stress distribution for a graded half plane with an edge crack and indented by a flat stamp as shown in Figure 2,  $a = 0$ ,  $\eta = 0.8$ ,  $\nu = 0.25$ ,  $b/d = 0.4$ .



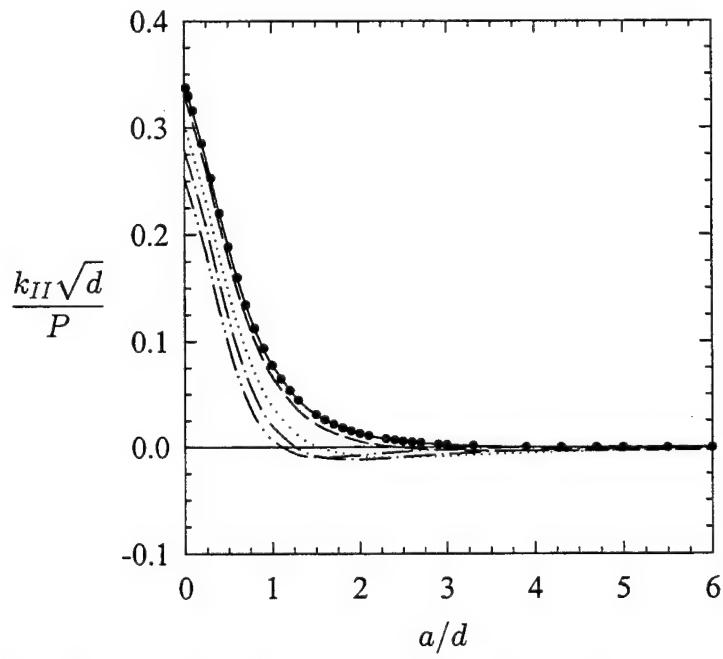
**Figure 21:** Mode I stress intensity factors for an edge crack in a graded half plane indented by a flat stamp as shown in Figure 2,  $(b - a)/d = 0.1$ ,  $\eta = 0$ ,  $\nu = 0.25$ .



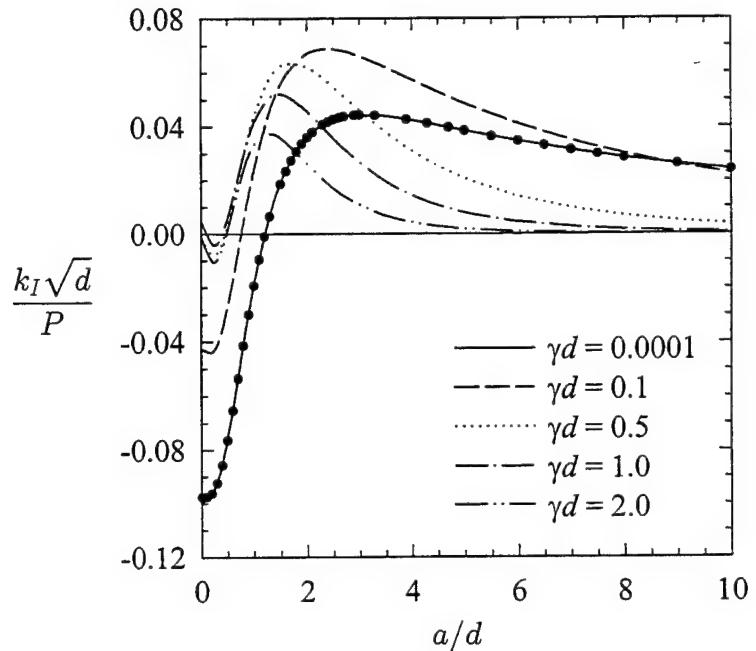
**Figure 22:** Mode II stress intensity factors for an edge crack in a graded half plane indented by a flat stamp as shown in Figure 2,  $(b - a)/d = 0.1$ ,  $\eta = 0$ ,  $\nu = 0.25$ .



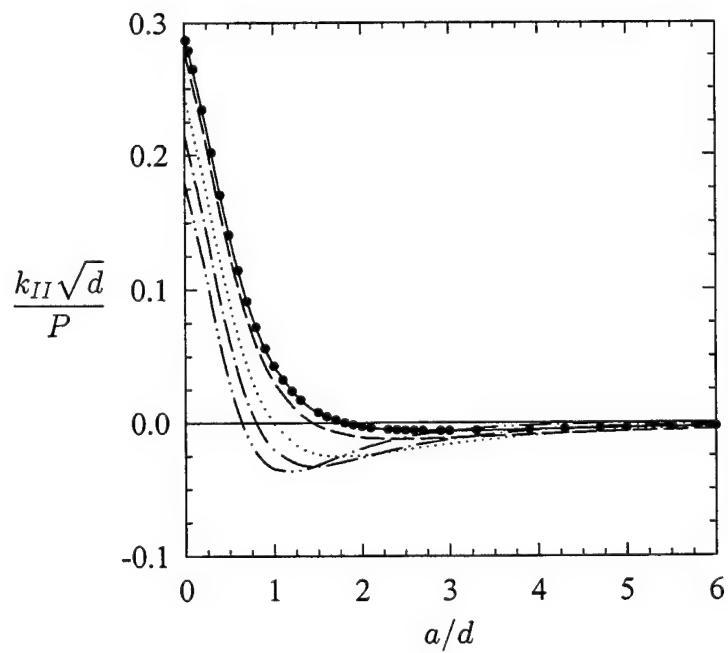
**Figure 23:** Mode I stress intensity factors for an edge crack in a graded half plane indented by a flat stamp as shown in Figure 2,  $(b - a)/d = 0.1$ ,  $\eta = 0.2$ ,  $\nu = 0.25$ .



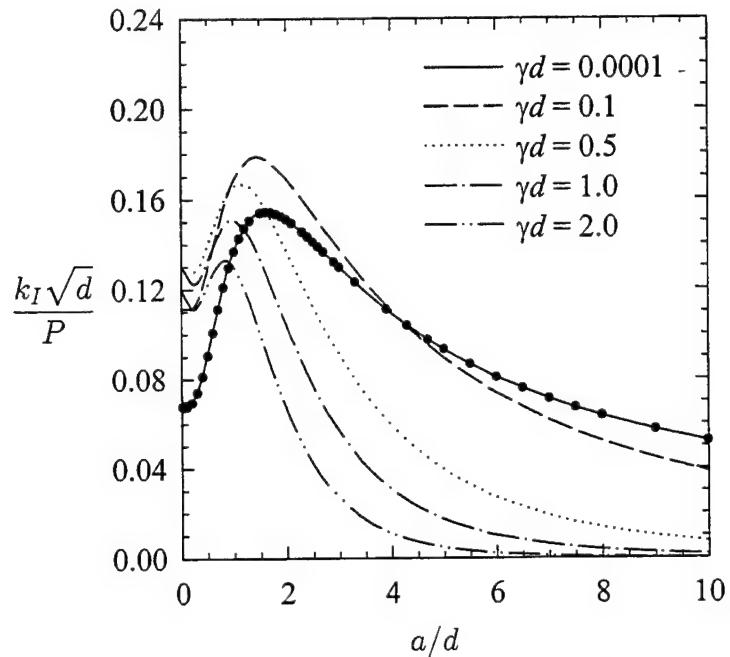
**Figure 24:** Mode II stress intensity factors for an edge crack in a graded half plane indented by a flat stamp as shown in Figure 2,  $(b - a)/d = 0.1$ ,  $\eta = 0.2$ ,  $\nu = 0.25$ .



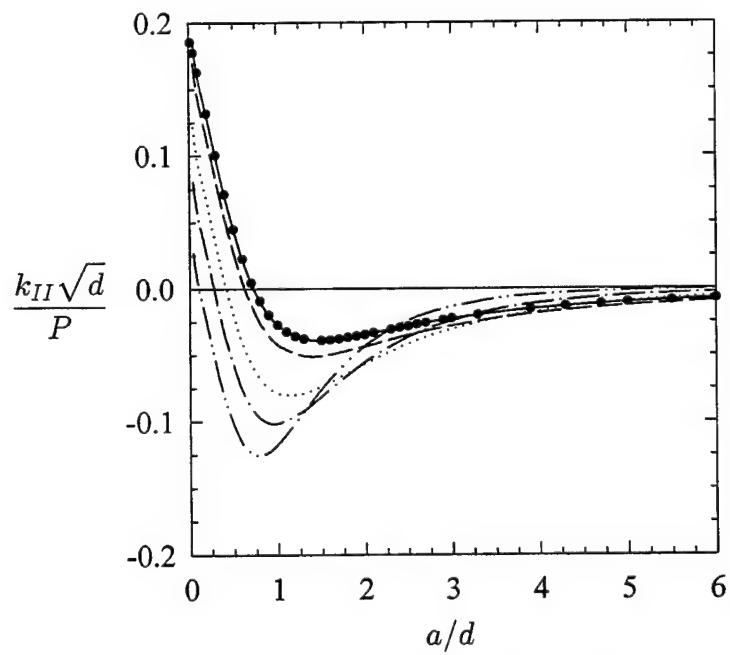
**Figure 25:** Mode I stress intensity factors for an edge crack in a graded half plane indented by a flat stamp as shown in Figure 2,  $(b - a)/d = 0.1$ ,  $\eta = 0.4$ ,  $\nu = 0.25$ .



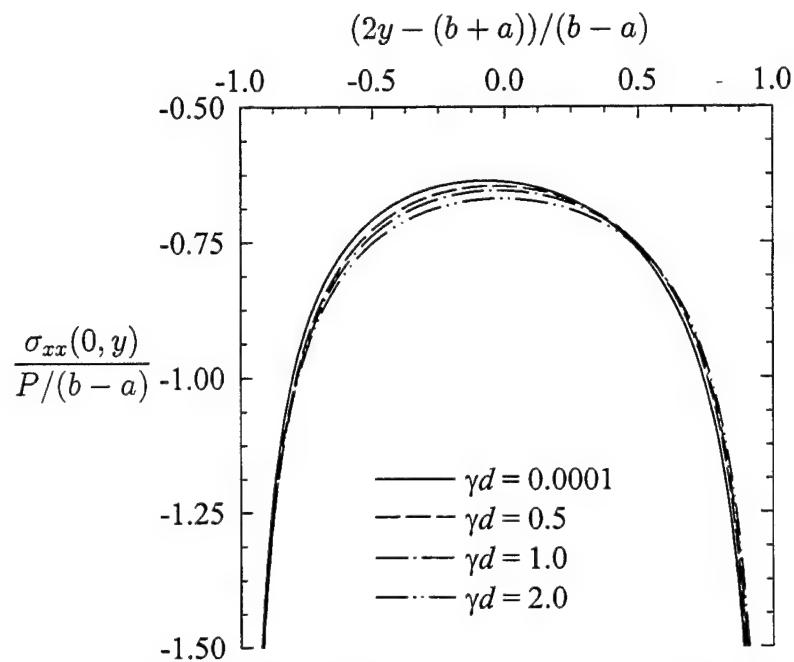
**Figure 26:** Mode II stress intensity factors for an edge crack in a graded half plane indented by a flat stamp as shown in Figure 2,  $(b - a)/d = 0.1$ ,  $\eta = 0.4$ ,  $\nu = 0.25$ .



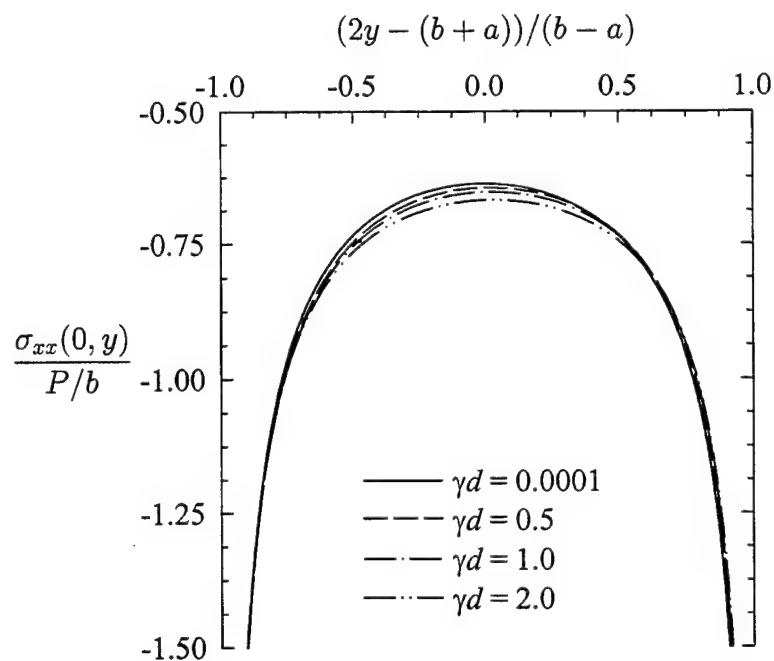
**Figure 27:** Mode I stress intensity factors for an edge crack in a graded half plane indented by a flat stamp as shown in Figure 2,  $(b - a)/d = 0.1$ ,  $\eta = 0.8$ ,  $\nu = 0.25$ .



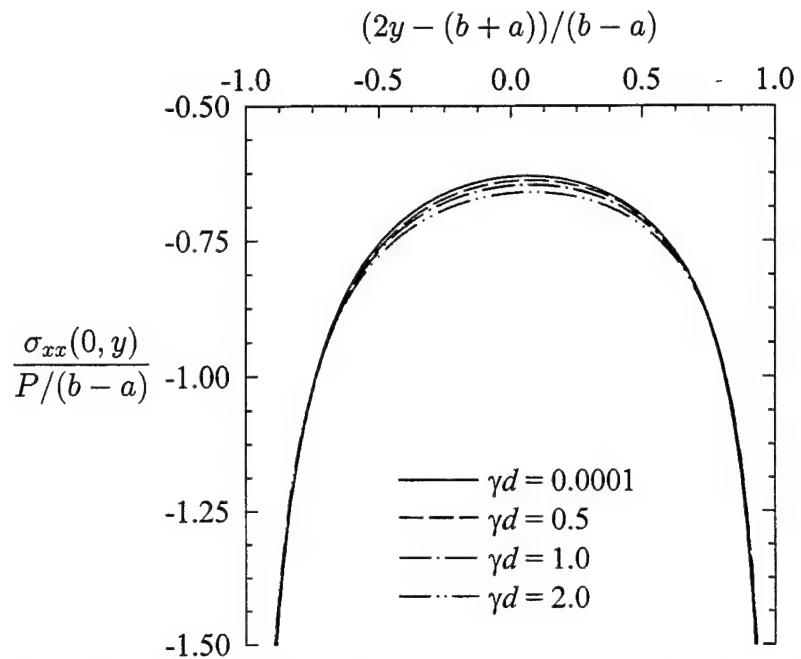
**Figure 28:** Mode II stress intensity factors for an edge crack in a graded half plane indented by a flat stamp as shown in Figure 2,  $(b - a)/d = 0.1$ ,  $\eta = 0.8$ ,  $\nu = 0.25$ .



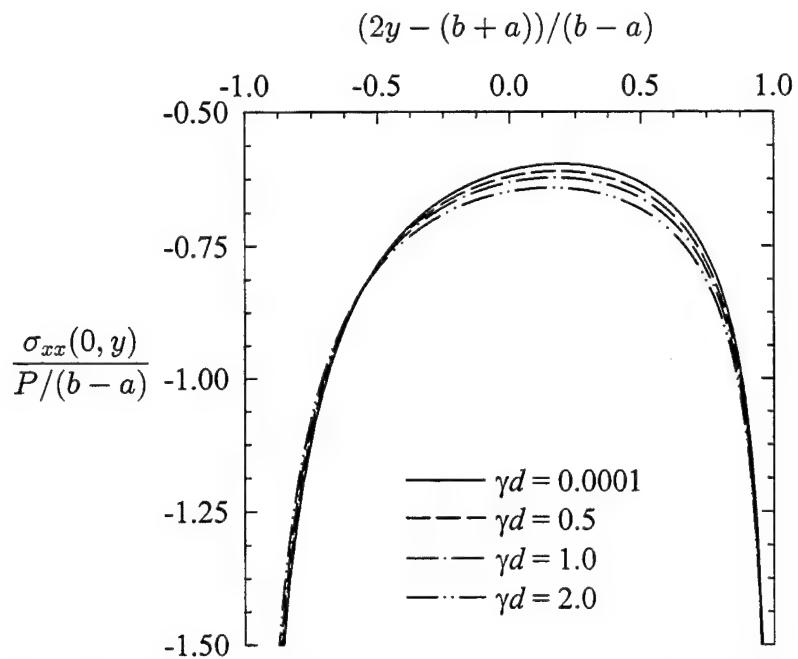
**Figure 29:** Contact stress distribution for a graded half plane with an edge crack and indented by a flat stamp as shown in Figure 2,  $(b - a)/d = 0.1$ ,  $\eta = 0$ ,  $\nu = 0.25$ ,  $a/d = 0.4$ .



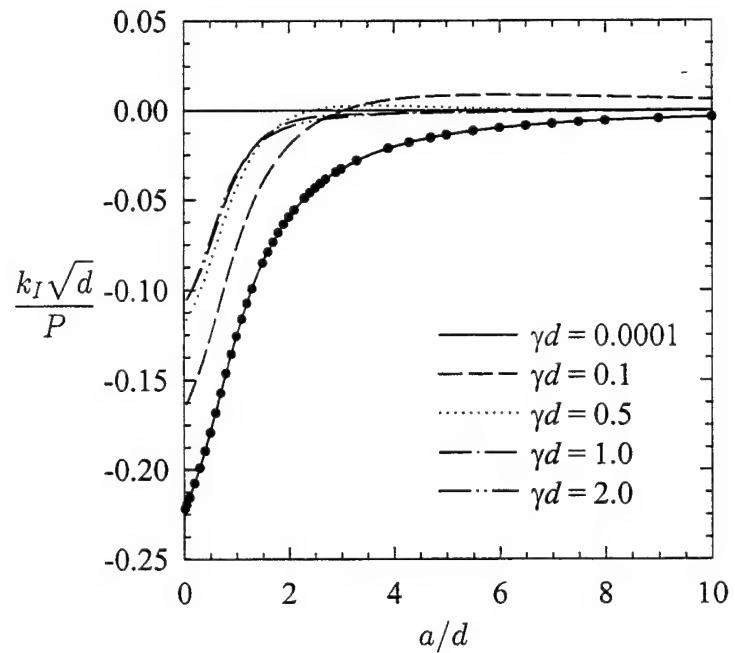
**Figure 30:** Contact stress distribution for a graded half plane with an edge crack and indented by a flat stamp as shown in Figure 2,  $(b - a)/d = 0.1$ ,  $\eta = 0.2$ ,  $\nu = 0.25$ ,  $a/d = 0.4$ .



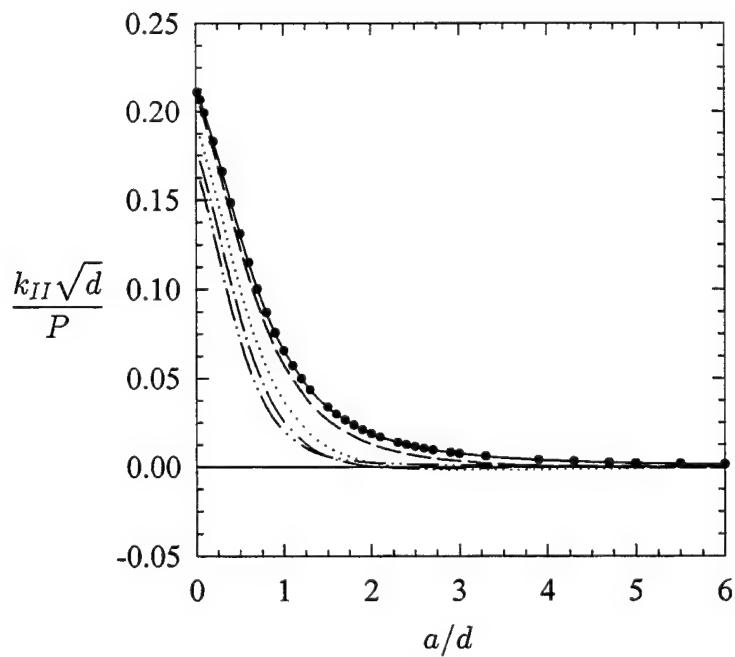
**Figure 31:** Contact stress distribution for a graded half plane with an edge crack and indented by a flat stamp as shown in Figure 2,  $(b - a)/d = 0.1$ ,  $\eta = 0.4$ ,  $\nu = 0.25$ ,  $a/d = 0.4$ .



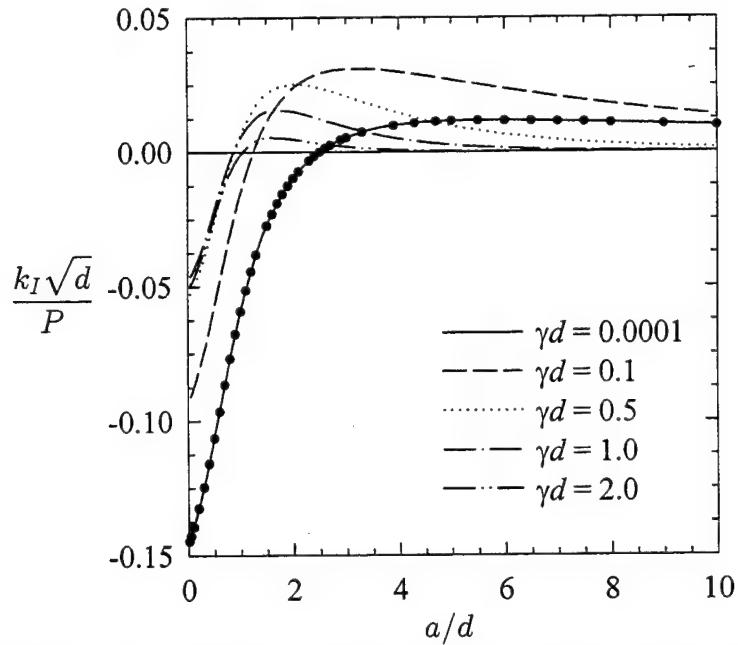
**Figure 32:** Contact stress distribution for a graded half plane with an edge crack and indented by a flat stamp as shown in Figure 2,  $(b - a)/d = 0.1$ ,  $\eta = 0.8$ ,  $\nu = 0.25$ ,  $a/d = 0.4$ .



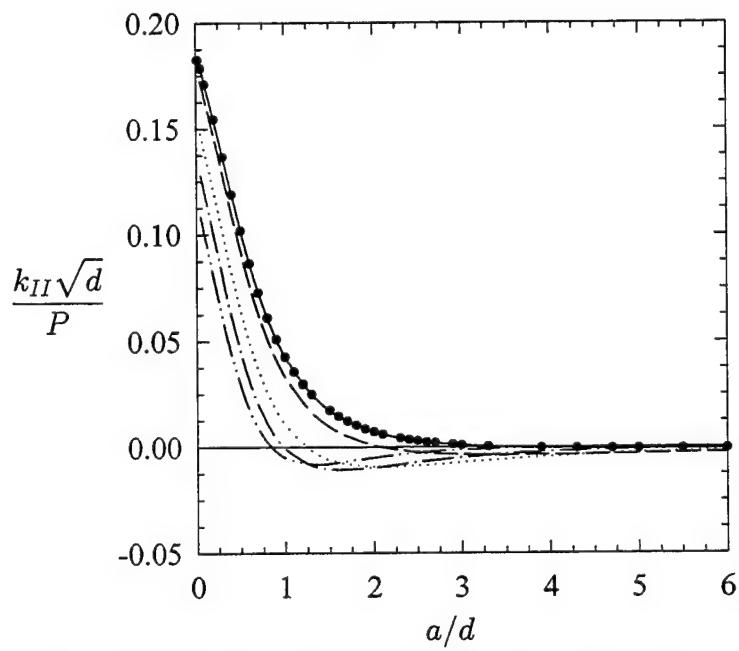
**Figure 33:** Mode I stress intensity factors for an edge crack in a graded half plane indented by a flat stamp as shown in Figure 2,  $(b - a)/d = 1.0$ ,  $\eta = 0$ ,  $\nu = 0.25$ .



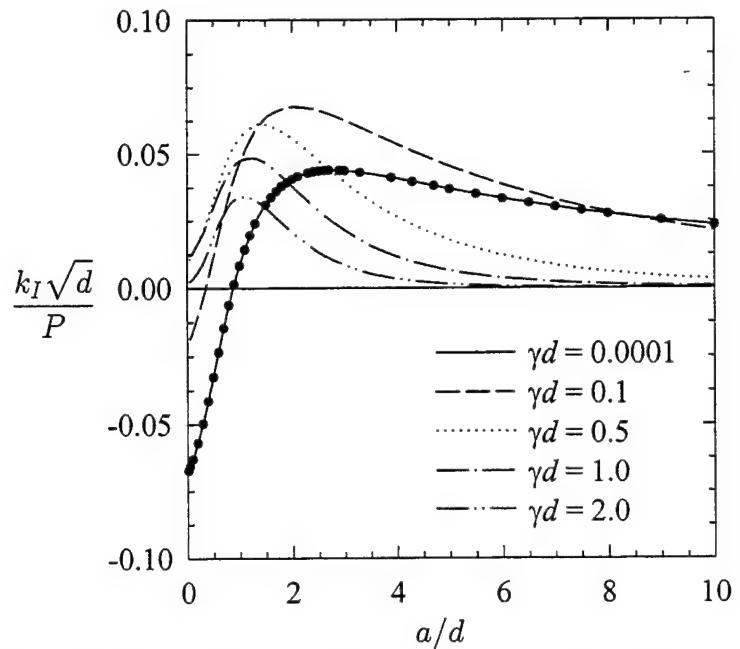
**Figure 34:** Mode II stress intensity factors for an edge crack in a graded half plane indented by a flat stamp as shown in Figure 2,  $(b - a)/d = 1.0$ ,  $\eta = 0$ ,  $\nu = 0.25$ .



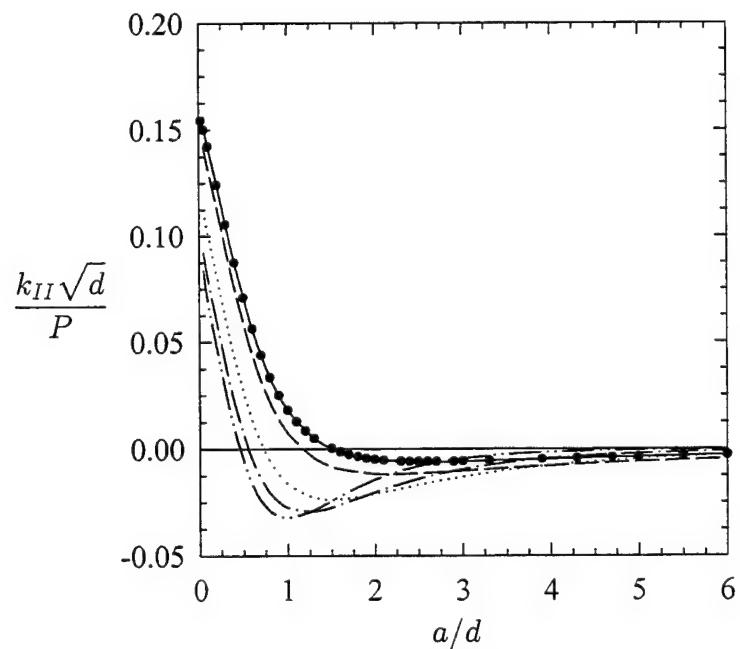
**Figure 35:** Mode I stress intensity factors for an edge crack in a graded half plane indented by a flat stamp as shown in Figure 2,  $(b - a)/d = 1.0$ ,  $\eta = 0.2$ ,  $\nu = 0.25$ .



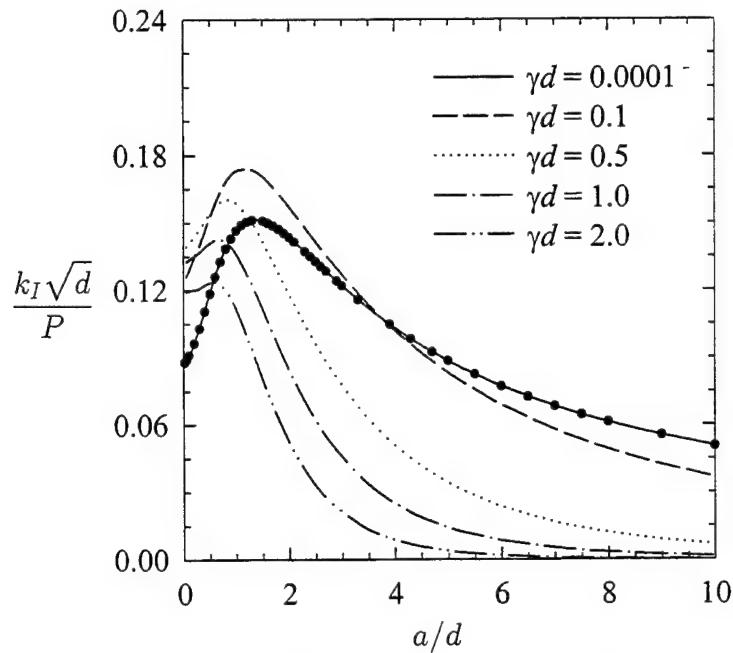
**Figure 36:** Mode II stress intensity factors for an edge crack in a graded half plane indented by a flat stamp as shown in Figure 2,  $(b - a)/d = 1.0$ ,  $\eta = 0.2$ ,  $\nu = 0.25$ .



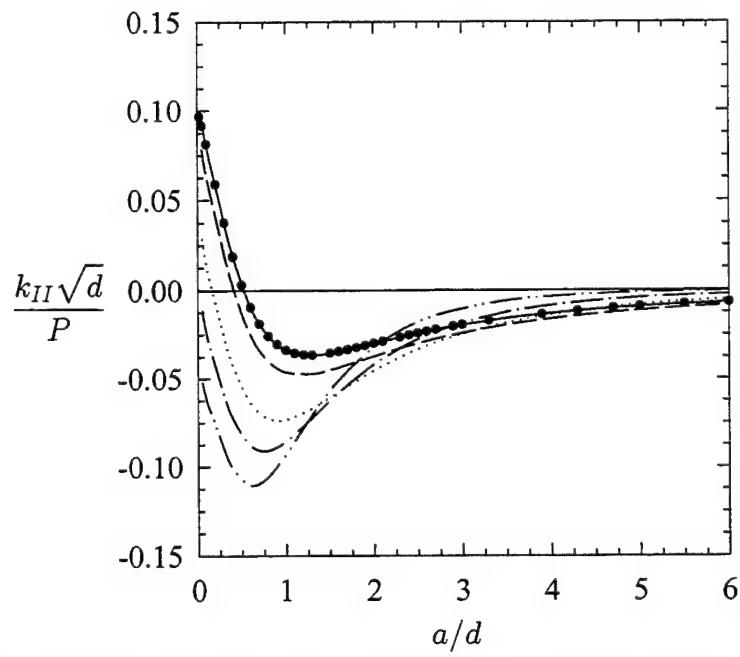
**Figure 37:** Mode I stress intensity factors for an edge crack in a graded half plane indented by a flat stamp as shown in Figure 2,  $(b - a)/d = 1.0$ ,  $\eta = 0.4$ ,  $\nu = 0.25$ .



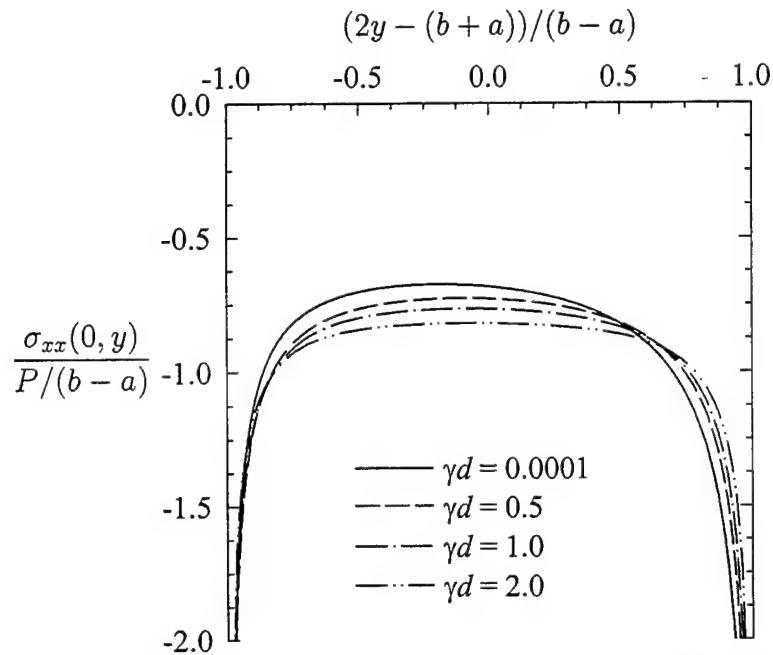
**Figure 38:** Mode II stress intensity factors for an edge crack in a graded half plane indented by a flat stamp as shown in Figure 2,  $(b - a)/d = 1.0$ ,  $\eta = 0.4$ ,  $\nu = 0.25$ .



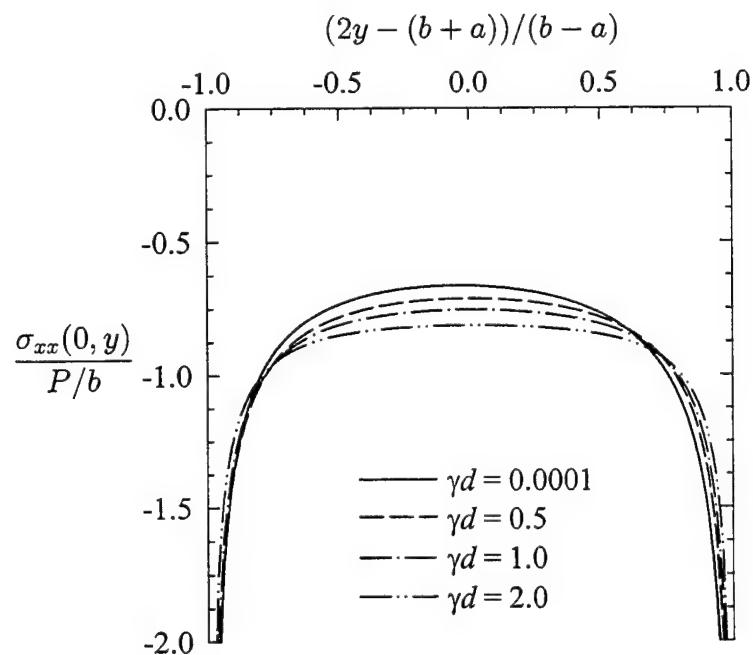
**Figure 39:** Mode I stress intensity factors for an edge crack in a graded half plane indented by a flat stamp as shown in Figure 2,  $(b - a)/d = 1.0$ ,  $\eta = 0.8$ ,  $\nu = 0.25$ .



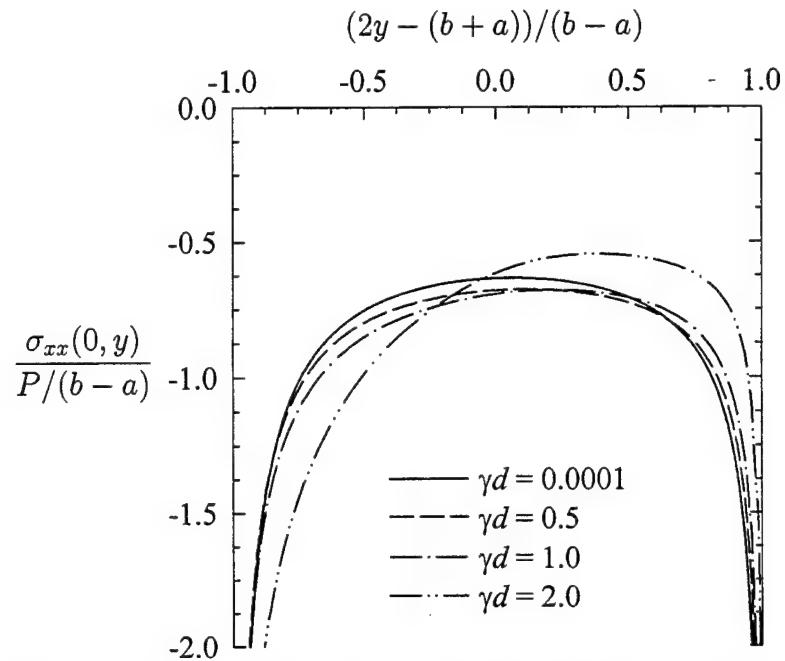
**Figure 40:** Mode II stress intensity factors for an edge crack in a graded half plane indented by a flat stamp as shown in Figure 2,  $(b - a)/d = 1.0$ ,  $\eta = 0.8$ ,  $\nu = 0.25$ .



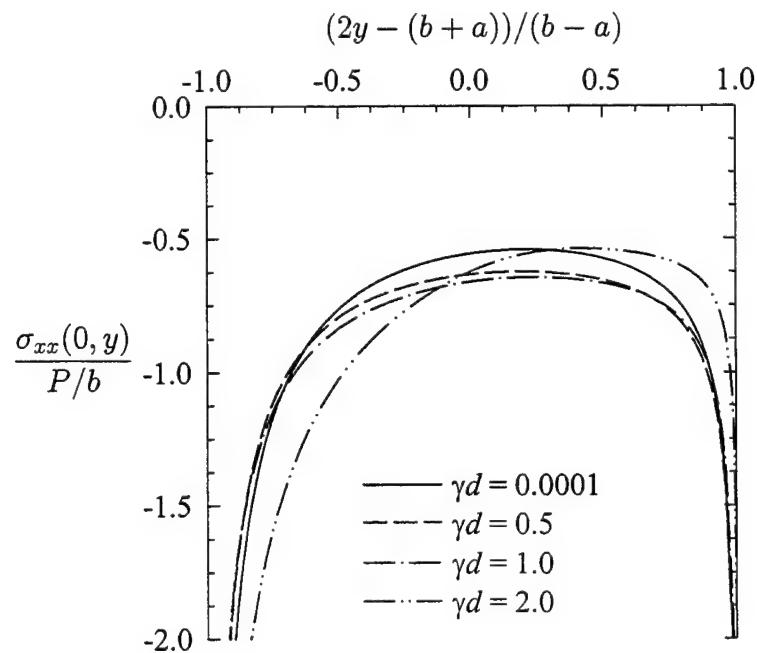
**Figure 41:** Contact stress distribution for a graded half plane with an edge crack and indented by a flat stamp as shown in Figure 2,  $(b - a)/d = 1.0$ ,  $\eta = 0$ ,  $\nu = 0.25$ ,  $a/d = 0.4$ .



**Figure 42:** Contact stress distribution for a graded half plane with an edge crack and indented by a flat stamp as shown in Figure 2,  $(b - a)/d = 1.0$ ,  $\eta = 0.2$ ,  $\nu = 0.25$ ,  $a/d = 0.4$ .



**Figure 43:** Contact stress distribution for a graded half plane with an edge crack and indented by a flat stamp as shown in Figure 2,  $(b - a)/d = 1.0$ ,  $\eta = 0.4$ ,  $\nu = 0.25$ ,  $a/d = 0.4$ .



**Figure 44:** Contact stress distribution for a graded half plane with an edge crack and indented by a flat stamp as shown in Figure 2,  $(b - a)/d = 1.0$ ,  $\eta = 0.8$ ,  $\nu = 0.25$ ,  $a/d = 0.4$ .

## APPENDIX A

### Some Useful Integrals

Here, we give the expressions that are used to evaluate the integrals involving the asymptotic expansions of the integrands of the kernels. There are three types of integrals involving the asymptotic expansions. The expressions for each type are given below.

#### *Integrals of type 1*

In this case, the integrals that we want to evaluate are in the following form,

$$C_n = \int_A^\infty \frac{1}{\rho^n} \cos(\rho u) d\rho, \quad n = 1, 2, 3, \dots, N, \quad (\text{F1a})$$

$$S_n = \int_A^\infty \frac{1}{\rho^n} \sin(\rho u) d\rho, \quad n = 1, 2, 3, \dots, N. \quad (\text{F1b})$$

For  $n = 1$ , following results are obtained using MAPLE,

$$C_1 = -\text{Ci}(A|u|), \quad (\text{F2a})$$

$$S_1 = \text{sign}(u) \left( \frac{\pi}{2} - \text{Si}(A|u|) \right), \quad (\text{F2b})$$

where  $\text{Ci}()$  and  $\text{Si}()$  are cosine and sine integrals, respectively, and they are defined by

$$\text{Ci}(x) = \gamma_0 + \ln(x) + \int_0^x \frac{\cos(\alpha) - 1}{\alpha} d\alpha, \quad (\text{F3a})$$

$$\text{Si}(x) = \int_0^x \frac{\sin(\alpha)}{\alpha} d\alpha. \quad (\text{F3b})$$

and the Euler constant is  $\gamma_0 = 0.5772156649$ . For  $n > 1$ , integrating (F1a) and (F1b) by parts the following general recursive relations are obtained:

$$C_n = -\frac{1}{1-n} \frac{\cos(uA)}{A^{n-1}} + \frac{u}{1-n} S_{n-1}, \quad n > 1 \quad (\text{F4a})$$

$$S_n = -\frac{1}{1-n} \frac{\sin(uA)}{A^{n-1}} - \frac{u}{1-n} C_{n-1}, \quad n > 1. \quad (\text{F4b})$$

Following result is also used in the integration of asymptotic expressions,

$$\int_0^\infty \frac{\sin(\rho u)}{\rho} d\rho = \frac{\pi}{2} \text{sign}(u). \quad (\text{F5})$$

### *Integrals of type 2*

In this case, we consider the following integral:

$$R_n = \int_A^\infty \frac{1}{\rho^n} \exp(\rho u) d\rho, \quad n = 1, 2, 3, \dots, N, \quad [u < 0]. \quad (\text{F6})$$

For  $n = 1$ , following result is obtained using MAPLE,

$$R_1 = E_1(-Au), \quad E_1(z) = -\text{Ei}(-z), \quad (\text{F7a,b})$$

where  $\text{Ei}()$  is the exponential integral function. For  $n \geq 1$ , following expression is used which is given by Gradshteyn and Ryzhik [10]:

$$\int_A^\infty \frac{\exp(-px)}{x^{n+1}} dx = (-1)^{n+1} \frac{p^n \text{Ei}(-pA)}{n!} + \frac{\exp(-pA)}{A^n} \sum_{k=0}^{n-1} \frac{(-1)^k p^k A^k}{n(n-1)\dots(n-k)},$$

$$[p > 0], \quad (\text{F8})$$

(F8) reduces to (F7) for  $n = 1$ .

### *Integrals of type 3*

Type 3 integrals are in the following form:

$$C_n = \int_A^\infty \frac{1}{\rho^n} \cos(\rho v) \exp(\rho u) d\rho, \quad n = 1, 2, 3, \dots, N, \quad [u < 0]. \quad (\text{F9a})$$

$$S_n = \int_A^\infty \frac{1}{\rho^n} \sin(\rho v) \exp(\rho u) d\rho, \quad n = 1, 2, 3, \dots, N, \quad [u < 0]. \quad (\text{F9b})$$

(F9) are evaluated using the following expressions which are given by Gradshteyn and Ryzhik [10]:

$$\begin{aligned} \int_A^\infty x^{\mu-1} \exp(-\beta x) \cos(\delta x) dx &= \frac{1}{2} (\beta + i\delta)^{-\mu} \Gamma(\mu, (\beta + i\delta)A) \\ &+ \frac{1}{2} (\beta - i\delta)^{-\mu} \Gamma(\mu, (\beta - i\delta)A), \quad [\Re(\beta) > |\Im(\delta)|], \end{aligned} \quad (\text{F10a})$$

$$\begin{aligned} \int_A^\infty x^{\mu-1} \exp(-\beta x) \sin(\delta x) dx &= \frac{i}{2} (\beta + i\delta)^{-\mu} \Gamma(\mu, (\beta + i\delta)A) \\ &- \frac{i}{2} (\beta - i\delta)^{-\mu} \Gamma(\mu, (\beta - i\delta)A), \quad [\Re(\beta) > |\Im(\delta)|], \end{aligned} \quad (\text{F10b})$$

where  $\Gamma(,)$  is the incomplete Gamma function. Following result is also used in the integration of asymptotic expressions,

$$\int_0^\infty \frac{1}{\rho} \sin(\rho v) \exp(\rho u) d\rho = -\arctan\left(\frac{v}{u}\right), \quad u < 0. \quad (\text{F11})$$

## APPENDIX B

### Some leading terms in asymptotic expansions

Here, we give the leading terms of asymptotic expansions  $K_{311}^{(i)\infty}(\omega, y)$ ,  $K_{312}^{(i)\infty}(\omega, y)$  and  $K_{321}^{(i)\infty}(\omega, y)$ ,  $K_{322}^{(i)\infty}(\omega, y)$  which are given by equations (129) and (147), respectively.

$$g_{11}^* = -2 \frac{\cos(\gamma y/2)(\exp(-\delta_1 y) - 1)\exp(\delta_1 y/2)}{(1 + \kappa)\delta_1}, \quad (\text{G1a})$$

$$g_{21}^* = 2 \frac{\sin(\gamma y/2)(\exp(-\delta_1 y) - 1)\exp(\delta_1 y/2)}{(1 + \kappa)\delta_1}, \quad (\text{G1b})$$

$$l_{11}^* = -2 \frac{\sin(\gamma y/2)(\exp(-\delta_1 y) - 1)\exp(\delta_1 y/2)}{(1 + \kappa)\delta_1}, \quad (\text{G1c})$$

$$l_{21}^* = -2 \frac{\cos(\gamma y/2)(\exp(-\delta_1 y) - 1)\exp(\delta_1 y/2)}{(1 + \kappa)\delta_1}. \quad (\text{G1d})$$

## APPENDIX C

### Kernels of the integral equations

In this Appendix, we give the transformed form of the kernels that are used in equations (182a,c) and the terms that are used in equations (185).

$$H_{11}(s_1, r) = \frac{d}{2}(H_{11s}(s_1, r) + H_{11f}(s_1, r)), \quad (\text{H1a})$$

$$H_{13}(s_1, r) = \frac{b-a}{2}(H_{13s}(s_1, r) + H_{13f}(s_1, r)), \quad (\text{H1b})$$

$$H_{22}(s_2, r) = \frac{d}{2}(H_{22s}(s_2, r) + H_{22f}(s_2, r)), \quad (\text{H1c})$$

$$H_{23}(s_1, r) = \frac{b-a}{2}(H_{23s}(s_2, r) + H_{13f}(s_2, r)), \quad (\text{H1d})$$

$$H_{31}(s_3, r) = \frac{d}{2}(H_{31s}(s_3, r) + H_{31f}(s_3, r)), \quad (\text{H1e})$$

$$H_{32}(s_3, r) = \frac{d}{2}(H_{32s}(s_3, r) + H_{32f}(s_3, r)), \quad (\text{H1f})$$

$$H_{33}(s_3, r) = \frac{b-a}{2}H_{33f}(s_3, r), \quad (\text{H1g})$$

where,

$$H_{ijs}(s_i, r) = h_{ijs}(x, t), \quad (\text{H2a})$$

$$H_{ijf}(s_i, r) = h_{ijf}(x, t), \quad (\text{H2b})$$

$$x = \begin{cases} \frac{d}{2}s_i + \frac{d}{2}, & i = 1, 2 \\ \frac{b-a}{2}s_i + \frac{b+a}{2}, & i = 3 \end{cases} \quad (\text{H3a})$$

$$t = \begin{cases} \frac{d}{2}r + \frac{d}{2}, & j = 1, 2 \\ \frac{b-a}{2}r + \frac{b+a}{2}, & j = 3 \end{cases} \quad (\text{H3b})$$

The terms used in equations (185) are given below:

$$\begin{aligned} m_{11n}(s_1) = & - \frac{2^{\alpha_1-1/2}\Gamma(-1/2)\Gamma(n+\alpha_1+1)}{\pi\Gamma(n+1/2+\alpha_1)} \times \\ & \times F\left(n+1, -n+1/2-\alpha_1; 3/2; (1-s_1)/2\right) \\ & + \int_{-1}^1 (1-r)^{-1/2} (1+r)^{\alpha_1} P_n^{(-1/2, \alpha_1)}(r) H_{11}(s_1, r) dr, \end{aligned} \quad (\text{H4a})$$

$$m_{13n}(s_1) = \int_{-1}^1 (1-r)^\beta (1+r)^{\alpha_2} P_n^{(\beta, \alpha_2)}(r) H_{13}(s_1, r) dr, \quad (\text{H4b})$$

$$\begin{aligned} m_{22n}(s_2) = & - \frac{2^{\alpha_1-1/2}\Gamma(-1/2)\Gamma(n+\alpha_1+1)}{\pi\Gamma(n+1/2+\alpha_1)} \times \\ & \times F\left(n+1, -n+1/2-\alpha_1; 3/2; (1-s_2)/2\right) + \\ & + \int_{-1}^1 (1-r)^{-1/2} (1+r)^{\alpha_1} P_n^{(-1/2, \alpha_1)}(r) H_{11}(s_2, r) dr, \end{aligned} \quad (\text{H4c})$$

$$m_{23n}(s_2) = \int_{-1}^1 (1-r)^\beta (1+r)^{\alpha_2} P_n^{(\beta, \alpha_2)}(r) H_{13}(s_2, r) dr, \quad (\text{H4d})$$

$$m_{31n}(s_3) = \int_{-1}^1 (1-r)^{-1/2} (1+r)^{\alpha_1} P_n^{(-1/2, \alpha_1)}(r) H_{31}(s_3, r) dr, \quad (\text{H4e})$$

$$m_{32n}(s_3) = \int_{-1}^1 (1-r)^{-1/2} (1+r)^{\alpha_1} P_n^{(-1/2, \alpha_1)}(r) H_{31}(s_3, r) dr, \quad (\text{H4f})$$

$$\begin{aligned}
m_{33n}(s_3) &= \frac{2^{\beta+\alpha_2}\Gamma(\beta)\Gamma(n+\alpha_2+1)}{\pi\Gamma(n+\beta+\alpha_2+1)} \times \\
&\quad \times F\left(n+1, -n-\beta-\alpha_2; 1-\beta; (1-s_3)/2\right) + \\
&\quad + \int_{-1}^1 (1-r)^\beta (1+r)^{\alpha_2} P_n^{(\beta, \alpha_2)}(r) H_{33}(s_3, r) dr,
\end{aligned} \tag{H4e}$$

Note that if  $\alpha_2 + \beta = -1, 0$ , or  $1$  (H4e) reduces to

$$m_{33n}(s_3) = \frac{2^{(\alpha_2+\beta)}}{\sin(\pi\beta)} P_{n+(\alpha_2+\beta)}^{(-\beta, -\alpha_2)}(s_3) + \int_{-1}^1 (1-r)^\beta (1+r)^{\alpha_2} P_n^{(\beta, \alpha_2)}(r) H_{33}(s_3, r) dr. \tag{H5a}$$

In this case, if  $(\alpha_2 + \beta) = -1$  and  $n = 0$  we have

$$m_{330}(s_3) = \int_{-1}^1 (1-r)^\beta (1+r)^{\alpha_2} H_{33}(s_3, r) dr. \tag{H5b}$$